AFCRC-TN-58-589
ASTIA DOCUMENT NO. AD 207520

NEW YORK UNIVERSITY
Institute of Mathematical Sciences
Division of Electromagnetic Research

RESEARCH REPORT No. EM-119

How Dark is the Shadow of a Round-Ended Screen?

JOSEPH B. KELLER

Contract No. AF 19(604)1717
OCTOBER, 1958

EM-119

Reprinted from THE JOURNAL OF THE ACOUSTICAL SOCIETY OF AMERICA, Vol. 36, No. 1, 32–40, January, 1964

Geometrical Theory of Elastic Surface-Wave Excitation and Propagation

JOSEPH B. KELLER AND FRANK C. KARAL, JR.

Courant Institute of Mathematical Sciences, New York University, New York, New York
(Received 27 September 1963)

A geometrical theory is devised for the description and calculation of surface waves, such as Rayleigh waves, on boundaries or interfaces of elastic solids. It applies to curved surfaces and to inhomogeneous media. The theory, which is an extension of geometrical optics, involves complex rays that travel from the source to the surface, then along the surface, and finally from the surface to points in the solid. It also includes phases and amplitudes associated with each point on each ray. Geometrical formulas are derived for the determination of these phases and amplitudes. They involve certain excitation and radiation coefficients, which are also determined. The total displacement at a point is the sum of the displacements on all the rays through the point, each of which is constructed from the corresponding phase and amplitude. The theory applies to periodic waves of high frequency and short wavelength, as well as to the rapidly varying portions of any waveform. It is a generalization of the authors' earlier geometrical theory of scalar surface waves [J. Appl. Phys. 31, 1039 (1960)].

INTRODUCTION

A SURFACE wave is a wave that travels along a surface near which the wavemotion is confined, e.g., a Rayleigh wave. Because of the difficulty of solving exactly elasticity problems involving such waves, we have devised a geometrical method for their approximate solution. It is an extension of our theory of scalar surface waves[1] and is based upon geometrical optics. Therefore, it also provides a geometrical picture of how surface waves propagate. It is valid for periodic waves of high frequency and short wavelength and for the rapidly varying portions of any waveform. However, experience with other problems of wave propagation shows that it is accurate even for wavelengths as large as other dimensions in the problem.

Geometrical optics determines the rays along which waves travel. It also yields the phase or arrival time at each point, and can be used to find the amplitude and direction of the wavemotion. This leads to an expression for an elastic wavemotion that is the first term in an infinite series. This has been shown by V. M. Babich and A. S. Alekseev[2] and by us, and we have shown how to determine the subsequent terms.[3] However, although this series improves the description of the motion on the usual ray, it fails to describe diffracted and surface waves. An additional series is required for each such wave. We have shown how to construct the leading term in each series by an extension of geometrical optics, called the "geometrical theory of diffraction."[4] That theory has been applied successfully to various scalar-, acoustic-, and electromagnetic-diffraction problems, and F. Gilbert[5] has applied it to certain elastic-diffraction problems. Now, we show how to apply it to elastic surface waves.

In the next section, we introduce various new rays. In Sec. II, we determine the phase and amplitude of the wavemotion on each ray in terms of certain excitation and radiation coefficients. These coefficients are proportional to each other, as we show in Sec. III by means of the reciprocity principle. In Sec. IV, we apply the method to determine the Rayleigh wave produced on

[1] J. B. Keller and F. C. Karal, Jr., "Surface Wave Excitation and Propagation," J. Appl. Phys. 31, 1039–1046 (1960).

[2] V. M. Babich and A. S. Alekseev, "The Ray Method for Calculating the Intensity of Wave Fronts," Bull. Acad. Sci. USSR, Geophys. Ser. No. 1, 9–15 (1958).

[3] F. C. Karal, Jr., and J. B. Keller, "Elastic Wave Propagation in Homogeneous and Inhomogeneous Media," J. Acoust. Soc. Am. 31, 694–705 (1959).

[4] J. B. Keller, "A Geometrical Theory of Diffraction," in *Proceedings of Symposia in Applied Mathematics, Volume 8: Calculus of Variations and Its Applications*, edited by L. M. Graves (McGraw-Hill Book Co., Inc., New York, 1958).

[5] F. Gilbert, "Scattering of Impulsive Elastic Waves by a Smooth Convex Cylinder," J. Acoust. Soc. Am. 32, 841–857 (1960).

the plane surface on an elastic half-space by a compressional line source. By comparing the result with the known short-wavelength asymptotic form of the exact solution of the problem, we determine the excitation coefficient. This coefficient can then be used for any wave incident on any surface. All the preceding results apply to time-harmonic waves. In Sec. V, we transform them to apply to transient waves.

The theory, which can be applied easily to various problems, yields the first term in a series. Further terms can be derived by methods that have been used in similar scalar and electromagnetic problems.

I. COMPLEX RAYS

With each point $\mathbf{x}$ of an isotropic elastic medium, there are associated two propagation velocities $V_c(\mathbf{x})$ and $V_s(\mathbf{x})$, called the compressional and shear velocities. In terms of each velocity, we can define rays in space, called compressional and shear rays, by means of Fermat's principle or the differential equations that follow from it. Each ray can be represented parametrically as a function $\mathbf{x}(s)$ of the arclength s. We also define complex rays of each type as complex valued functions $\mathbf{x}(s)$ that satisfy the ray differential equations or that are determined by Fermat's principle applied to complex curves. This definition is possible, provided the velocities are defined for complex values of $\mathbf{x}$, which is the case if they are analytic functions. Then, the complex rays are curves in a three-dimensional complex space. Of course, our primary interest in them concerns the points where they cross the real space. In unbounded media, the complex rays that start from a point in the real space and then leave this space usually never return to it. But, as we see here, in bounded media the complex rays may interact with the boundary and then return to the real space, in analogy with the fact that real rays can return to the vicinity of their source after reflection from a boundary.

Let us now consider a surface S along which a surface wave can propagate with the velocity $V_R(\mathbf{x})$. Real surface rays on S are curves that satisfy the differential equation for rays traveling on a surface with the velocity $V_R(\mathbf{x})$ or that are determined by the Fermat principle for curves on a surface. We define complex surface rays by applying Fermat's principle or the ray differential equations to complex curves on S. As before, this is possible if $V_R(\mathbf{x})$ and S are defined in the complex $\mathbf{x}$ space.

When a space ray hits a surface S it gives rise to two reflected rays—one shear and one compressional—and two transmitted rays (provided there is a medium on the "transmitted" side of S). The directions of these rays are determined by the appropriate laws of reflection and refraction. These laws are consequences of Fermat's principle applied to curves having a point on the surface. It follows from them that a real incident ray produces two real reflected rays; but one or both of the transmitted rays may be complex. The occurrence of complex transmitted rays is called total reflection and the waves associated with them are called evanescent waves.

If surface waves can propagate along S, we assume that the incident ray may also produce a surface ray. Then, the directions of the incident and surface rays are also related by the law of refraction in which the surface-ray velocity plays the role of the velocity along the transmitted ray. This is a consequence of applying Fermat's principle to curves having an arc on S on which the velocity is $V_R(\mathbf{x})$ and an arc in space. Since the surface ray lies in S, it follows from the law of refraction that the angle θ between the incident ray and the normal to S is given by

$$\sin\theta = V(\mathbf{x})/V_R(\mathbf{x}). \tag{1}$$

Here, $V(\mathbf{x})$ is either $V_c(\mathbf{x})$ or $V_s(\mathbf{x})$, according as the incident ray is a compressional or shear ray. This same relation holds when a space ray is shed by the surface ray, which we also assume to occur. In addition, the incident-ray direction, the surface-ray direction, and the surface normal must all lie in the same plane.

Surface waves are usually slow waves, which means that they travel slower than space waves and, consequently, $V(\mathbf{x})/V_R(\mathbf{x})>1$ at real points $\mathbf{x}$. But then, by Eq. (1), θ cannot be real at real points, so, in this case, a real ray cannot produce a real surface ray. This is why we have to introduce complex rays to construct a ray description of surface-wave propagation.

There are certain surface waves that travel with the space velocity on the convex side of the surface. We have previously called them diffracted surface waves and the associated rays diffracted surface rays. In isotropic elastic media, there are two kinds: diffracted compressional and shear surface rays. When the incident ray and the diffracted surface ray are of the same type, then $V(\mathbf{x})=V_R(\mathbf{x})$ and it follows from Eq. (1) that $\theta=\pi/2$. Thus, an incident ray of either kind tangent to the surface can produce a diffracted surface ray of the same kind, and the latter can shed space rays of the same kind along its tangents. When the space ray and the diffracted surface ray are of different types, then it follows from Eq. (1) that the diffracted surface ray must have the greater velocity, as has been pointed out by Gilbert.[5] We do not consider diffracted surface rays in the present work.

Let us now consider two points Q and P in an elastic medium bounded by a surface S along which a surface wave can propagate. There are direct and reflected rays from Q to P. In addition, a wave might travel from Q to P by going from Q to some point Q_1 on S, traveling along S from Q_1 to another point P_1, and then leaving S at P_1 to go to P (see Fig. 1). We suppose that this happens, and we want to determine Q_1 and P_1. The paths from Q to Q_1 and from P_1 to P are space rays, while that from Q_1 to P_1 is a surface ray. Since the rays QQ_1 and

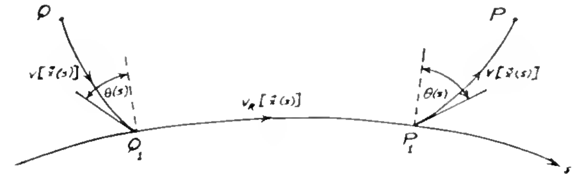


FIG. 1. The complex ray from Q reaches the surface at the complex point Q_1 where it produces a complex surface ray. This ray travels along the surface with the velocity $V_R[\mathbf{x}(s)]$ and sheds complex space rays. The ray that leaves the surface at P_1 passes through P. Since the rays QQ_1 and PP_1 may be either shear or compressional rays, there are four kinds of paths from Q to P and four pairs of points Q_1, P_1. Each pair Q_1, P_1 is determined by the condition that Eq. (1) must be satisfied at each point with the appropriate value of $V[\mathbf{x}(s)]$.

PP_1 may be either shear or compressional rays, there are four kinds of paths from Q to P and four pairs of points Q_1, P_1. Each pair Q_1, P_1 is determined by the condition that Eq. (1) must be satisfied at each point with the appropriate value of V.

II. PHASE AND AMPLITUDE

We now show how to use the rays to determine the elastic displacement $\mathbf{u}(P)$ at the point P. We first postulate that there is a displacement associated with each ray through P, and that the total displacement at P is the sum of the displacements on all the rays through P. We now assume that the motion is time-harmonic with angular frequency ω, and we write the displacement as the real part of $u(P)e^{-i\omega t}$, letting u be complex. Then, on each ray we write $u(P)=A(P)e^{i\varphi(P)}$, and we call $\varphi(P)$ the phase at P and $A(P)$ the amplitude at P. This representation of u is made unique by the requirement that $\varphi(P)$ be proportional to ω, while the amplitude A may be complex but may not contain any phase factor proportional to ω. Finally, we assume that $\varphi(P)$ is given by the relation

$$\varphi(P)=\varphi(P_1)+\omega\int_{P_1}^{P} ds/V[\mathbf{x}(s)]. \tag{2}$$

The integral in Eq. (2) is with respect to arclength s along the ray from P_1 to P and V is the velocity associated with the ray, which is either V_c, V_s, or V_R, depending upon the type of ray.

Let us apply Eq. (2) to each of the four paths from Q to P that contain arcs on S. We may write all four results together by introducing the subscripts i and j, each of which can take the two values c and s, denoting compressional and shear rays. Then, $\varphi_{ij}(P)$ denotes the phase at P on a ray whose first part, from Q to Q_1, is of type i and whose last part, from P_1 to P, is of type j. The values of Q_1 and P_1 on such a ray are denoted by Q_{ij} and P_{ij}. Then, we have

$$\varphi_{ij}(P)=\varphi_i(Q)+\omega\int_{Q}^{Q_{ij}} ds/V_i[\mathbf{x}(s)] \\ +\omega\int_{Q_{ij}}^{P_{ij}} ds/V_R[\mathbf{x}(s)]+\omega\int_{P_{ij}}^{P} ds/V_j[\mathbf{x}(s)]. \tag{3}$$

Here, $\varphi_i(Q)$ denotes the phase at Q on a ray of type i.

To determine the variation along a ray of the amplitude $A(P)=|\mathbf{A}(P)|$, we apply the principle of energy conservation to a tube of rays. For a space ray of type i, this principle states that $A^2\rho V_i d\tau$ is constant along the tube where ρ is the density of the medium and $d\tau$ is the cross-sectional area of the tube. This follows from Eqs. (115) and (139) of Ref. 3 for compressional rays and from Eq. (113) for shear rays. We assume that the same principle applies to a strip of surface rays in the form $A^2\rho' V_R d\sigma=\text{const}$, where $d\sigma$ denotes the width of the strip and ρ' is a density, depending upon the densities on the two sides of the surface.

To find the direction of $\mathbf{A}(P)$, we utilize the facts that on a compressional ray $\mathbf{A}$ is tangential and on a shear ray $\mathbf{A}$ is normal. Thus, in the compressional case, we write $\mathbf{A}_c=A_c\mathbf{k}$, where $\mathbf{k}$ is a unit vector tangent to the ray, while in the shear case we write $\mathbf{A}_s=A_s\mathbf{n}$, where $\mathbf{n}$ is the appropriate unit vector normal to the ray. Then, utilizing the energy principle that we have on the compressional and shear rays, respectively,

$$\mathbf{A}_c(P)=A_c(P_1)\left[\frac{d\tau(P_1)}{d\tau(P)}\frac{\rho(P_1)}{\rho(P)}\frac{V_c(P_1)}{V_c(P)}\right]^{\frac{1}{2}}\mathbf{k}(P), \tag{4}$$

$$\mathbf{A}_s(P)=A_s(P_1)\left[\frac{d\tau(P_1)}{d\tau(P)}\frac{\rho(P_1)}{\rho(P)}\frac{V_s(P_1)}{V_s(P)}\right]^{\frac{1}{2}}\mathbf{n}(P). \tag{5}$$

The variation of $\mathbf{n}(P)$ along the ray depends upon the torsion of the ray[1] and for a straight ray $\mathbf{n}(P)$ is constant. The direction of the amplitude associated with a surface ray is completely determined by the direction of the ray and the properties of the media at the surface and, therefore, we need not consider this direction. Thus, letting $\hat{A}$ denote the amplitude on a surface ray, we write merely

$$\hat{A}(P)=\hat{A}(P_1)\left[\frac{d\sigma(P_1)}{d\sigma(P)}\frac{\rho'(P_1)}{\rho'(P)}\frac{V_R(P_1)}{V_R(P)}\right]^{\frac{1}{2}}. \tag{6}$$

Let us now determine the amplitudes $\mathbf{A}_{ij}(P)$ on each of the four rays that go from Q to P by way of the surface S. To do so, we make use of Eqs. (4)–(6) and certain additional relations between the amplitudes on the surface and space rays at the point on the surface where they meet. Let $\mathbf{A}_c(Q_{cj})$ and $\mathbf{A}_s(Q_{sj})$ be the amplitudes on the incident compressional and shear rays, respectively, at the points Q_{cj} and Q_{sj}. Then, we assume that the amplitudes $\hat{A}_c(Q_{cj})$ and $\hat{A}_s(Q_{sj})$ on the resulting surface rays are proportional to the incident amplitudes. Thus, we write.

$$\hat{A}_c(Q_{cj})=E_c(Q_{cj})A_c(Q_{cj}), \tag{7}$$

$$\hat{A}_s(Q_{sj})=E_s(Q_{sj})\mathbf{A}_s(Q_{sj})\cdot\mathbf{N}(Q_{sj}). \tag{8}$$

In Eqs. (7) and (8), $E_c(Q_{cj})$ and $E_s(Q_{sj})$ are excitation coefficients that, we assume, depend only upon the properties of the media at Q_{cj}, respectively. In Eq. (8),

$\mathbf{N}(Q_{sj})$ denotes the unit normal to S at Q_{sj}. We have introduced it because only the component of displacement in the plane of incidence is effective in exciting the surface wave, and this is proportional to the normal component.

We next assume that the amplitude $\mathbf{A}_{ij}(P_{ij})$ on a ray leaving the surface is proportional to the amplitude $\hat{A}_i(P_{ij})$ on the surface ray. We express this proportionality in terms of the radiation coefficients $R_j(P_{ij})$, which depend upon the properties of the media at P_{ij}. Thus, we write

$$A_{ij}(P_{ij})=R_j(P_{ij})\hat{A}_i(P_{ij}). \tag{9}$$

When the final ray is a shear ray, $j=s$, we assume that the unit vector $\mathbf{n}(P_{is})$, which determines the direction of the displacement at P_{is}, lies in the plane containing the tangents to the surface ray and the shear ray at P_{is}. It is, of course, also normal to the shear ray.

Upon combining Eqs. (4)–(9), we obtain

$$A_{ij}(P)=\left[\frac{d\tau(P_{ij})}{d\tau(P)}\frac{\rho(P_{ij})}{\rho(P)}\frac{V_j(P_{ij})}{V_j(P)}\right]^{\frac{1}{2}}\times R_j(P_{ij})\left[\frac{d\sigma(Q_{ij})}{d\sigma(P_{ij})}\frac{\rho'(Q_{ij})}{\rho'(P_{ij})}\frac{V_R(Q_{ij})}{V_R(P_{ij})}\right]^{\frac{1}{2}}\times E_i(Q_{sj})A_i(Q_{ij}). \tag{10}$$

Let us also write the phase $\varphi_{ij}(P)$, given by Eq. (3), in the form

$$\varphi_{ij}(P)=\varphi_i(Q_{ij})+\omega\int_{Q_{ij}}^{P_{ij}}ds/V_R[\mathbf{x}(s)]+\omega\int_{P_{ij}}^{P}ds/V_j[\mathbf{x}(s)]. \tag{11}$$

Then, we may write the total displacement at P due to surface rays in the form

$$\mathbf{u}(P)=A_{cc}(P)\mathbf{k}_{cc}(P)e^{i\varphi_{cc}(P)}+A_{sc}(P)\mathbf{k}_{sc}(P)e^{i\varphi_{sc}(P)}+A_{cs}(P)\mathbf{n}_{cs}(P)e^{i\varphi_{cs}(P)}+A_{ss}(P)\mathbf{n}_{ss}(P)e^{i\varphi_{ss}(P)}. \tag{12}$$

Here, $\mathbf{k}_{cc}(P)$ and $\mathbf{k}_{sc}(P)$ are unit tangents to the compressional rays at P, while $\mathbf{n}_{cs}(P)$ and $\mathbf{n}_{ss}(P)$ are the appropriate unit normals to the shear rays at P.

Equation (12), together with (10) and (11), expresses the main result of our geometrical theory of elastic surface waves. It shows how to express the elastic displacement at P due to such waves as a sum of terms. There is one term for each ray through P that comes from a surface ray on S. In Eq. (12), we have written four terms, since usually there will be four rays through P if the source of the motion is a point. For some geometries and for other sources, there may be more rays and, therefore, more terms must be included in Eq. (12). Each term involves the amplitude and phase on the incident ray at the point where it produces the surface ray. It also involves the surface-ray velocity V_R, the density ρ', the two excitation coefficients E_c and E_s, and the two radiation coefficients R_c and R_s. These quantities must be determined by other considerations—for example, from the solutions of special or canonical problems. Once they are known, the comutation of $\mathbf{u}(P)$ is reduced to a geometrical calculation.

In the next section, we show that the excitation coefficients are expressible in terms of the radiation coefficients. In the subsequent sections, we determine these coefficients, as well as V_R and ρ', for a free surface by using the solution of a canonical problem. The same method can also be used for any other type of surface.

III. RECIPROCITY PRINCIPLE

There exists a relation between the excitation and radiation coefficients, as we now show by utilizing the reciprocity principle.[6] This principle may be formulated as follows:

$$u_x(P,Qy)=u_y(Q,Px). \tag{13}$$

Here, $u_x(P,Qy)$ denotes the x component of displacement at P due to a unit force in the y direction applied to the medium at Q. Similarly, $u_y(Q,Px)$ is the y component of displacement at Q due to a unit force in the x direction at P. We apply Eq. (13) to that part of the displacement $\mathbf{u}$ that involves surface waves, which we have determined in the preceding section.

It is simplest to apply Eq. (13) to a two-dimensional problem in which the media, the surface, and $\mathbf{u}$ are all independent of one coordinate, say z. The results will then apply only to such two-dimensional configurations. However, we assume that the excitation and radiation coefficients depend only upon properties of the surface and the media in the plane of incidence, at the points of excitation and radiation, respectively. Then, the two-dimensional results concerning these coefficients will apply in general.

On the basis of these considerations, we consider the displacement $\mathbf{u}$ produced by a unit force in the y direction at Q. (In three dimensions, this point force represents a line source through Q parallel to the z axis.) Then, the phase $\varphi_i(Q_{ij})$ can be expressed by Eq. (2) in the form

$$\varphi_i(Q_{ij})=\omega\int_Q^{Q_{ij}}ds/V_i[\mathbf{x}(s)]. \tag{14}$$

The amplitudes $\mathbf{A}_c(Q_{cj})$ and $\mathbf{A}_s(Q_{sj})$ can be expressed in the form Eqs. (4) and (5). However, since all the rays lie in planes $z=$ const, the areas $d\tau$ in these and the subsequent equations may be taken to be lengths in the xy plane multiplied by a unit length in the z direction. For the same reason, the lengths $d\sigma$ may be taken to

[6] Lord Rayleigh, *The Theory of Sound* (Macmillan and Co. Ltd., London, 1894; reprinted, Dover Publications, Inc., New York, 1945), Vol. 1, p. 153–155.

be constant lengths in the z direction, so that $d\sigma(Q_{ij})/d\sigma(P_{ij})=1$.

In applying Eqs. (4) and (5), we replace P by Q and choose P_1 to be a point near Q. But then, $d\tau(P_1)=sd\theta(Q)$, where s is the distance from Q to P_1 and $d\theta(Q)$ is the angle between the two rays bounding a tube. As P_1 tends to Q, s tends to zero and $A_i(P_1)$ becomes infinite in such a way that the following limit exists: $\lim_{s\to 0} s^{\frac{1}{2}}A_i(P_1)=A_i'(Q)$. For a line source consisting of a unit force per unit length in the x direction in an infinite homogeneous medium, $A_i'(Q)$ is given by the following expression,[7] in which we write $A_i^x(x)$ to indicate the direction of the force:

$$A_c^x(Q)=e^{i\pi/4}(8\pi\omega)^{-\frac{1}{2}}[V_c(Q)]^{-\frac{3}{2}}[\rho(Q)]^{-1}\sin\theta(Q), \quad (15)$$

$$A_s^x(Q)=e^{i\pi/4}(8\pi\omega)^{-\frac{1}{2}}[V_s(Q)]^{-\frac{3}{2}}[\rho(Q)]^{-1}\cos\theta(Q). \quad (16)$$

In Eqs. (15) and (16), $\theta(Q)$ is the angle that the ray makes with the positive y axis at Q. The unit vector $\mathbf{n}$ is oriented so that $\mathbf{k}\times\mathbf{n}$ points in the positive z direction. In accordance with the geometrical method, we assume that $A_i^x(Q)$ is also given by Eqs. (15) or (16) for inhomogeneous and bounded media. Now, Eqs. (4) and (5) yield with P_1 and P replaced by Q and Q_{ij}, respectively,

$$A_i^x(Q_{ij})=e^{i\pi/4}(8\pi\omega)^{-\frac{1}{2}}[V_i(Q)]^{-1} \times\left[\frac{d\theta(Q)}{d\tau(Q_{ij})}\frac{1}{\rho(Q)\rho(Q_{ij})V_i(Q_{ij})}\right]^{\frac{1}{2}} \begin{matrix}\sin\\ \cos\end{matrix}\,\theta_{ij}(Q). \quad (17)$$

Here, and in subsequent equations, sin is to be used for $i=c$ and cos for $i=s$.

Upon inserting Eq. (17) into Eq. (10), we obtain

$$A_{ij}^x(P)=e^{i\pi/4}(8\pi\omega)^{-\frac{1}{2}}\left[\frac{d\tau(P_{ij})}{d\tau(P)}\frac{\rho(P_{ij})}{\rho(P)}\frac{V_j(P_{ij})}{V_j(P)} \times\frac{\rho'(Q_{ij})}{\rho'(P_{ij})}\frac{V_R(Q_{ij})}{V_R(P_{ij})}\frac{d\theta(Q)}{d\tau(Q_{ij})}\cdot\frac{1}{\rho(Q)\rho(Q_{ij})V_i(Q_{ij})}\right]^{\frac{1}{2}} \times[V_i(Q)]^{-1}R_j(P_{ij})E_i(Q_{ij})\begin{matrix}\sin\\ \cos\end{matrix}\,\theta_{ij}(Q). \quad (18)$$

Inserting Eq. (14) into Eq. (11) yields

$$\varphi_{ij}(P)=\omega\int_Q^{Q_{ij}} ds/V_i+\omega\int_{Q_{ij}}^{P_{ij}} ds/V_R+\omega\int_{P_{ij}}^{P} ds/V_j. \quad (19)$$

When Eqs. (18) and (19) are used in Eq. (12), they yield $\mathbf{u}(P)$. The y component of $\mathbf{u}(P)$ is

$$u_y(P)=A_{cc}(P)\cos\theta_{cc}(P)e^{i\varphi_{cc}(P)} +A_{sc}(P)\cos\theta_{sc}(P)e^{i\varphi_{sc}(P)} +A_{cs}(P)\sin\theta_{cs}(P)e^{i\varphi_{cs}(Q)} +A_{ss}(P)\sin\theta_{ss}(P)e^{i\varphi_{ss}(P)}. \quad (20)$$

Here, θ_{ij} is the angle between $\mathbf{k}_{ij}$ and the y axis, and we have utilized the fact that $\mathbf{n}_{is}$ lies in the xy plane and is perpendicular to $\mathbf{k}_{is}$. Similarly, we find

$$u_x(Q)=A_{cc}(Q)[-\sin\theta_{cc}(Q)]e^{i\varphi_{cc}(Q)} +A_{sc}(Q)[-\sin\theta_{sc}(Q)]e^{i\varphi_{sc}(Q)} +A_{cs}(Q)[\cos\theta_{cs}(Q)]e^{i\varphi_{cs}(Q)} +A_{ss}(Q)[\cos\theta_{ss}(Q)]e^{i\varphi_{ss}(Q)}. \quad (21)$$

To obtain $u_x(Q)$, we use Eqs. (18) and (19) with P and Q interchanged, taking account of the fact that the rays from Q to P are also the rays from P to Q. We still let i denote the type of ray from Q_{ij} to Q and j denote the type of ray from P to P_{ij} so that the labeling of the points P_{ij} and Q_{ij} is unchanged. Then, we find from Eq. (19) that $\varphi_{ij}(P)=\varphi_{ji}(Q)$. Since $u_x(Q)$ results from a force in the y direction at P, the coefficient $A_i^y(P)$ is obtained from Eqs. (15) and (16) by replacing θ by $\theta-(\pi/2)$.

We now insert Eqs. (20) and (21) into Eq. (13). The resulting equation must hold for all points P and Q and all values of ω. Therefore, the coefficients of $e^{i\varphi_{ij}(Q)}$ and $e^{i\varphi_{ji}(P)}$ must be equal. This yields the four equations

$$A_{cc}(P)\cos\theta_{cc}(P)=-A_{cc}(Q)\sin\theta_{cc}(Q); \quad (22)$$

$$A_{sc}(P)\cos\theta_{sc}(P)=A_{cs}(Q)\cos\theta_{cs}(Q); \quad (23)$$

$$A_{cs}(P)\sin\theta_{cs}(P)=-A_{sc}(Q)\sin\theta_{sc}(Q); \quad (24)$$

$$A_{ss}(P)\sin\theta_{ss}(P)=A_{ss}(Q)\cos\theta_{ss}(Q). \quad (25)$$

In these equations, we now use Eq. (18) and the corresponding expression with P and Q interchanged, noting that, in interchanging P and Q, $\theta_{ij}(Q)$ must be replaced by $\theta_{ij}(P)-(\pi/2)$. We then find that the trigonometric factors and the minus signs in these equations cancel out and that they all become of the form $A_{ij}(P)=A_{ji}(Q)$, with the sines and cosines omitted. We now use Eq. (18) for $A_{ij}(P)$ and for $A_{ji}(Q)$, equate these two expressions omitting the trigonometric factors, and transpose all factors involving P and P_{ij} to one side and those involving Q and Q_{ij} to the other. After simplifying, we can write the resulting equation in the form

$$\left[\left\{\frac{d\tau(P_{ij})}{d\tau(P)}\right\}_Q\left\{\frac{d\tau(P_{ij})}{d\theta(P)}\right\}_P\right]^{\frac{1}{2}} \times\frac{V_j(P_{ij})\rho(P_{ij})[V_j(P)]^{\frac{1}{2}}}{V_R(P_{ij})\rho'(P_{ij})}\frac{R_j(P_{ij})}{E_j(P_{ij})}$$
$$=\left[\left\{\frac{d\tau(Q_{ij})}{d\tau(Q)}\right\}_P\left\{\frac{d\tau(Q_{ij})}{d\theta(Q)}\right\}_Q\right]^{\frac{1}{2}} \times\frac{V_i(Q_{ij})\rho(Q_{ij})[V_i(Q)]^{\frac{1}{2}}}{V_R(Q_{ij})\rho'(Q_{ij})}\frac{R_i(Q_{ij})}{E_i(Q_{ij})}. \quad (26)$$

In Eq. (26), the subscript Q or P on a geometrical ratio

[7] H. Lamb, "On the Propagation of Tremors over the Surface of an Elastic Solid," Phil. Trans. Royal Soc. (London) **203**, 1–42 (1904).

indicates the source of the rays to be used in calculating that ratio.

The geometrical ratios in Eq. (26) can be calculated simply by letting Q approach Q_{ij} so that the ray from Q to Q_{ij} is essentially straight, and similarly for P and P_{ij}. Then, $\{d\tau(Q_{ij})/d\theta(Q)\}_Q = d(Q_{ij},Q)$ is the distance from Q_{ij} to Q, which will be complex if Q_{ij} is complex. Furthermore, from Ref. 1, Eq. (10), we find when the rays are straight that

$$\left\{\frac{d\tau(Q_{ij})}{d\tau(Q)}\right\}_P = [1+(\kappa+\dot{\theta}_i)d(Q_{ij},Q)\sec\theta_i]_{Q_{ij}}^{-1}. \quad (27)$$

Here, κ denotes the curvature of the surface at Q_{ij}, θ_i is the value of θ given by Eq. (1) at Q_{ij} with $V = V_i(Q_{ij})$, and $\dot{\theta}_i$ is the derivative of θ_i with respect to arclength along the surface. From Eq. (27) and the preceding ratio, we have

$$\left\{\frac{d\tau(Q_{ij})}{d\tau(Q)}\right\}_P \left\{\frac{d\tau(Q_{ij})}{d\theta(Q)}\right\}_Q = [\kappa_0+(\kappa+\dot{\theta}_i)\sec\theta_i]_{Q_{ij}}^{-1}. \quad (28)$$

Here, $\kappa_0 = 1/d(Q_{ij},Q)$ is the curvature of the wavefront incident at Q_{ij}.

We now use Eq. (28) on the right side of Eq. (26), and on the left side we insert the corresponding expression with P's and Q's interchanged and with θ_j in place of θ_i. Then, the left side of Eq. (26) is found to be independent of Q and the right side independent of P. Therefore, each side is a constant, which we shall write as a^{-2}. Furthermore, since we have let P approach P_{ij}, we may replace P by P_{ij} on the left side and similarly Q by Q_{ij} on the right. Upon equating the left side of Eq. (26) to a^{-2}, we obtain an equation for $E_j(P_{ij})$ that yields

$$E_j(P_{ij}) = a^2[\kappa_0+(\kappa+\dot{\theta}_j)\sec\theta_j]_{P_{ij}}^{-\frac{1}{2}}[V_j(P_{ij})]^{\frac{1}{2}}$$
$$\times R_j(P_{ij})\rho(P_{ij})/V_R(P_{ij})\rho'(P_{ij}). \quad (29)$$

It is convenient to introduce the new radiation coefficient $R_j'(P_{ij})$ defined by

$$R_j'(P_{ij}) = aR_j(P_{ij})[V_j(P_{ij})]^{\frac{1}{4}}$$
$$\times[\rho'(P_{ij})V_R(P_{ij})]^{-\frac{1}{2}}[\rho(P_{ij})]^{\frac{1}{2}}. \quad (30)$$

Then, Eq. (29) becomes

$$E_j(P_{ij}) = a[\kappa_0+(\kappa+\dot{\theta}_j)\sec\theta_j]_{P_{ij}}^{-\frac{1}{2}}$$
$$\times\frac{[V_j(P_{ij})]^{\frac{1}{4}}}{[V_R(P_{ij})]^{\frac{1}{2}}}\left[\frac{\rho(P_{ij})}{\rho'(P_{ij})}\right]^{\frac{1}{2}}. \quad (31)$$

This is the relationship between E_j and R_j that we have been seeking. We see from Eq. (26) that it also holds when P_{ij} is replaced by Q_{ij}.

Although Eq. (31) has been derived in the special case of cylindrical symmetry (i.e., for the two-dimensional case), we assume that it holds in general, provided that κ_0 and κ are the curvatures of the incident wavefront and the surface in the plane of incidence. This is the plane containing the tangent to the incident ray and the normal to the surface. In addition, $\dot{\theta}_j$ is the derivative of θ_j along the surface ray.

Let us now use Eq. (31) for $E_i(Q_{ij})$ and eliminate E_i from Eq. (10). In this way, we obtain

$$A_{ij}(P) = A_i(Q_{ij})R_j'(P_{ij})R_i(Q_{ij})$$
$$\times\left[\frac{V_i(Q_{ij})}{V_j(P_{ij})}\right]^{\frac{1}{4}}\left[\frac{V_i(Q_{ij})\rho(Q_{ij})}{V_j(P)\rho(P)}\frac{d\sigma(Q_{ij})}{d\sigma(P_{ij})}\frac{d\tau(P_{ij})}{d\tau(P)}\right]^{\frac{1}{2}}$$
$$\cdot[\kappa_0+(\kappa+\dot{\theta}_i)\sec\theta_i]_{Q_{ij}}^{-\frac{1}{2}}. \quad (32)$$

When Eq. (32) is used in Eq. (12) ,it yields the surface-wave field at P due to an arbitrary wave incident upon the surface. We note that Eq. (32) does not involve ρ', which has been absorbed into the radiation coefficient R_i'. In the next section, we determine R_i' for a free surface, using a method that can also be applied to other kinds of surfaces. Once R_i' is known, Eqs. (12) and (32) will express the surface-wave field in terms of known geometrical quantities and material properties.

For a homogeneous medium bounded by a plane surface, the space rays and surface rays are straight lines. Then, Eq. (3) yields, with $\varphi_i(Q) = 0$,

$$\varphi_{ij}(P) = \omega d(Q,Q_{ij})/V_i + \omega d(Q_{ij},P_{ij})/V_R$$
$$+\omega d(P_{ij},P)/V_j. \quad (33)$$

In the two-dimensional case of a homogeneous medium, Eq. (32) becomes

$$A_{ij}(P) = R_j'R_i'(V_i/V_j)^{\frac{3}{4}}[\kappa_0(Q_{ij})]^{-\frac{1}{2}}A_i(Q_{ij}). \quad (34)$$

In Eq. (34), we have made use of the facts that $\dot{\theta}_i = 0$, $\kappa = 0$, $d\sigma(Q_{ij})/d\sigma(P_{ij}) = 1$, $d\tau(P_{ij})/d\tau(P) = 1$ from Eq. (27), and all coefficients are constants. If the field is produced by a line source at Q, it follows from Eqs. (4) and (5) that $[\kappa_0(Q_{ij})]^{-\frac{1}{2}}A_i(Q_{ij})$ does not depend upon the distance from Q to Q_{ij} but depends only upon the direction $\theta_{ij}(Q)$ of the ray from Q to Q_{ij}. Calling this product $A_i'[\theta_{ij}(Q)]$, we may write Eq. (34) as

$$A_{ij}(P) = R_j'R_i'(V_i/V_j)^{\frac{3}{4}}A_i'[\theta_{ij}(Q)]. \quad (35)$$

The quantity $A_i'[\theta_{ij}(Q)]$ expresses the strength and angular pattern or radiation pattern of the source and depends upon the source characteristics. In the case of a unit force per unit length in the x direction, A_i' is given by Eqs. (15) and (16). Upon using Eqs. (21) and (22) in Eq. (12), we obtain $\mathbf{u}(P)$ due to a line source at Q in a semi-infinite homogeneous medium.

IV. DETERMINATION OF THE RADIATION COEFFICIENT

We now use the preceding results to determine the field $\mathbf{u}(P)$ due to a compressional line source at Q in a semi-infinite homogeneous medium bounded by a free surface. Then, we compare the results with the exact

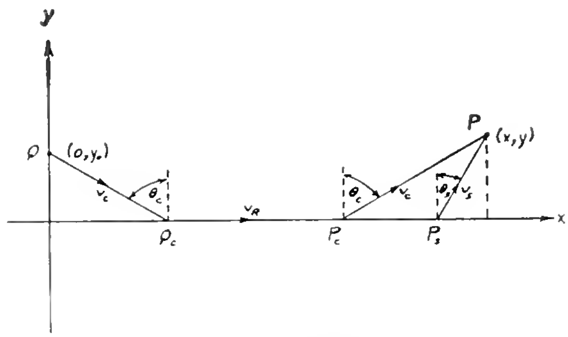


FIG. 2. An elastic medium with a free surface at $y=0$ occupies the half-space $y\geq 0$. The compressional line source is parallel to the z axis and intersects the plane $z=0$ at Q with coordinates $x=0$, $y=y_0$. There are two ray paths from the source to the observation point at P and are denoted by QQ_cP_cP and QQ_cP_sP.

solution of the equations of elasticity for this case, when that solution is expanded asymptotically for large values of the frequency ω. This comparison will show that the two results are identical when the radiation coefficients R_i' are chosen appropriately. We then assume that these coefficients have the values so determined in all cases involving a free surface, provided the material constants in the expressions for R_i' are given the values appropriate to the point of excitation or radiation.

We begin by observing that the point Q_{ij} depends only upon i and Q, while the point P_{ij} depends only upon j and P. This is because the ray from Q to Q_{ij} is a straight line that makes an angle with the normal to the surface determined by Eq. (1). In Eq. (1), only V_i occurs, so the angle depends on V_i, and, therefore, the location of Q_{ij} depends only upon V_i and the location of Q. The same considerations apply to P_{ij}. Therefore, we write Q_i instead of Q_{ij} and P_j instead of P_{ij}.

Let the medium occupy the half-space $y>0$ with a free surface at $y=0$. Let the line source be parallel to the z axis and let it intersect the plane $z=0$ at Q with coordinates $x=0$, $y=y_0$. Then, the coordinates of Q_c are $(y_0\tan\theta_c, 0)$. If the observation point P is at (x,y) with $x>0$, then P_c is at $(x-y\tan\theta_c,\ 0)$ and P_s is at $(x-y\tan\theta_s,\ 0)$ (see Fig. 2). Between these points, we find the distances to be

$$\begin{aligned} d(Q,Q_c)&=y_0\sec\theta_c;\\ d(Q_c,P_c)&=x-(y_0+y)\tan\theta_c; \qquad (36)\\ d(P_c,P)&=y\sec\theta_c;\\ d(Q_c,P_s)&=x-y_0\tan\theta_c-y\tan\theta_s; \qquad (37)\\ d(P_s,P)&=y\sec\theta_s. \end{aligned}$$

Upon using Eqs. (36) and (37) in (33), we obtain

$$\varphi_{cc}(P)=\omega V_c^{-1}[|x|\sin\theta_c+(y_0+y)\cos\theta_c]; \tag{38}$$

$$\varphi_{cs}(P)=\omega V_c^{-1}[|x|\sin\theta_c+y_0\cos\theta_c+V_c^{-1}V_sy\cos\theta_s]. \tag{39}$$

Here, we have used the relations $V_s=V_R\sin\theta_s$ and $V_c=V_R\sin\theta_c$, which follow from Eq. (1). We have also written $|x|$ instead of x, so the results will be valid for $x<0$. Since we consider a compressional source, $A_s'(Q)=0$, so we needn't consider the shear rays from Q.

Let us now use Eqs. (38) and (39), together with Eqs. (35) and (12), to obtain for the surface wave at P

$$\begin{aligned}\mathbf{u}(P)=&A_c'(\theta_c)R_c'^2\mathbf{k}_c\exp\{i\omega V_c^{-1}[|x|\sin\theta_c\\ &+(y_0+y)\cos\theta_c]\}+A_c'(\theta_c)R_c'R_s'(V_c/V_s)^{\frac{1}{2}}\mathbf{n}_s\\ &\times\exp\{i\omega V_c^{-1}[|x|\sin\theta_c+y_0\cos\theta_c\\ &+V_c^{-1}V_sy\cos\theta_s]\}.\end{aligned} \tag{40}$$

The unit vectors $\mathbf{k}_c$ and $\mathbf{n}_s$ are

$$\mathbf{k}_c=(\sin\theta_c,\cos\theta_c), \tag{41}$$

$$\mathbf{n}_s=(\cos\theta_s,-\sin\theta_s). \tag{42}$$

The result (40) involves, in addition to known geometrical quantities, the three unknown quantities $A_c'(\theta_c)$, R_c', and R_s'. Let us consider the exact solution for the displacement in an infinite homogeneous medium due to a compressional line source of unit strength (unit volume per unit length). This solution, when expanded for large values of ω, yields

$$A_c'(\theta_c)=(\omega/8\pi V_c)^{\frac{1}{2}}e^{i3\pi/4}. \tag{43}$$

Next, we consider the exact solution for the displacement due to the same source in a semi-infinite homogeneous medium with a free surface. This solution has been expanded asymptotically for large values of ω. In the notation of the previous paragraphs of this section, the surface wave at P is found to be

$$\begin{aligned}\mathbf{u}(x,y)\sim&\mathbf{k}_c(\omega/V_c)e^{i\pi/2}(2\sin^2\theta_c-V_c^2V_s^{-2})^2[\partial G(\theta_c)/\partial\theta]^{-1}\\ &\times\exp\{i\omega V_c^{-1}[|x|\sin\theta_c+(y_0+y)\cos\theta_c]\}\\ &+\mathbf{n}_s(\omega/V_s)e^{i\pi/2}2\sin\theta_c(2\sin^2\theta_c-V_c^2V_s^{-2})\\ &\times(V_c^2V_s^{-2}-\sin^2\theta_c)^{\frac{1}{2}}[\partial G(\theta_c)/\partial\theta]^{-1}\\ &\times\exp\{i\omega V_c^{-1}[|x|\sin\theta_c+y_0\cos\theta_c\\ &+V_c^{-1}V_sy\cos\theta_s]\}.\end{aligned} \tag{44}$$

In Eq. (44), the function $G(\theta)$ is defined by

$$\begin{aligned}G(\theta)=&(2\sin^2\theta-V_c^2V_s^{-2})^2+4\sin^2\theta\\ &\times\cos\theta(V_c^2V_s^{-2}-\sin^2\theta)^{\frac{1}{2}}.\end{aligned} \tag{45}$$

Upon comparing Eq. (44) with Eq. (40), we find that they agree exactly, provided that

$$\begin{aligned}A_c'(\theta_c)R_c'^2=&e^{i\pi/2}(\omega/V_c)(2\sin^2\theta_c-V_c^2V_s^{-2})^2\\ &\times[\partial G(\theta_c)/\partial\theta]^{-1},\end{aligned} \tag{46}$$

$$\begin{aligned}A_c'(\theta_c)R_c'R_s'(V_c/V_s)^{\frac{1}{2}}=&2e^{i\pi/2}(\omega/V_s)\\ &\times\sin\theta_c(2\sin^2\theta_c-V_c^2V_s^{-2})(V_c^2V_s^{-2}-\sin^2\theta_c)^{\frac{1}{2}}\\ &\times[\partial G(\theta_c)/\partial\theta]^{-1}.\end{aligned} \tag{47}$$

When Eq. (43) is used for $A_c'(\theta_c)$ in Eqs. (46) and (47), they can be solved for R_c' and R_s'. The results are

$$\begin{aligned}R_c'=&(8\pi\omega/V_c)^{\frac{1}{4}}e^{-i\pi/8}(2\sin^2\theta_c-V_c^2V_s^{-2})\\ &\times[\partial G(\theta_c)/\partial\theta]^{-\frac{1}{2}},\end{aligned} \tag{48}$$

$$R_s'=(8\pi\omega/V_c)^{\frac{1}{2}}e^{-i\pi/8}2\sin\theta_c\cos\theta_c \times[\partial G(\theta_c)/\partial\theta]^{-\frac{1}{2}}(V_c/V_s)^{\frac{1}{2}}. \quad (49)$$

The above equations determine the radiation coefficients.

Let us now insert Eqs. (48), (49), and (32) into Eq. (12) and replace the amplitudes $A_c(Q_{cj})$ and $A_s(Q_{sj})$ by Eqs. (4) and (5). This yields the final result for the surface wave excited on a curved free surface by an arbitrary incident field in an inhomogeneous elastic medium. It is

$$\mathbf{u}(P)=\sqrt{\pi}e^{-i(\pi/4)}\omega^{\frac{1}{2}}[\mathbf{B}_{cc}(P)e^{i\omega\varphi_{cc}'(P)}+\mathbf{B}_{sc}(P)e^{i\omega\varphi_{sc}'(P)} +\mathbf{B}_{cs}(P)e^{i\omega\varphi_{cs}'(P)}+\mathbf{B}_{ss}e^{\omega\varphi_{ss}'(P)}], \quad (50)$$

where

$$\mathbf{B}_{cc}(P)=2(2)^{\frac{1}{2}}A_c(Q_{cc})[V_c(P_{cc})]^{-\frac{1}{2}}\left[\frac{V_c(Q_{cc})\rho(Q_{cc})d\sigma(Q_{cc})d\tau(P_{cc})}{V_c(P)\rho(P)d\sigma(P_{cc})d\tau(P)}\right]^{\frac{1}{2}}\cdot[2\sin^2\theta_c-V_c^2V_s^{-2}]_{P_{cc}}$$
$$\times[2\sin^2\theta_c-V_c^2V_s^{-2}]_{Q_{cc}}\left[\frac{\partial G(Q_c)}{\partial\theta}\right]_{P_{cc}}^{-\frac{1}{2}}\left[\frac{\partial G(\theta_c)}{\partial\theta}\right]_{Q_{cc}}^{-\frac{1}{2}}\cdot[\kappa_0+(\kappa+\dot\theta_c)]_{Q_{cc}}^{-\frac{1}{2}}\mathbf{k}_{cc}(P); \quad (51)$$

$$\mathbf{B}_{sc}(P)=2(2)^{\frac{1}{2}}A_s(Q_{sc})[V_c(P_{sc})]^{-\frac{1}{2}}\left[\frac{V_c(Q_{sc})}{V_s(Q_{sc})}\right]^{5/4}\left[\frac{V_s(Q_{sc})\rho(Q_{sc})d\sigma(Q_{sc})d\tau(P_{sc})}{V_c(P)\rho(P)d\sigma(P_{sc})d\tau(P)}\right]^{\frac{1}{2}}\cdot[2\sin^2\theta_c-V_c^2V_s^{-2}]_{P_{sc}}$$
$$\times[2\sin\theta_c\cos\theta_c]_{Q_{sc}}\left[\frac{\partial G(\theta_c)}{\partial\theta}\right]_{P_{sc}}^{-\frac{1}{2}}\left[\frac{\partial G(\theta_c)}{\partial\theta}\right]_{Q_{sc}}^{-\frac{1}{2}}\cdot[\kappa_0+(\kappa+\dot\theta_s)]_{Q_{sc}}^{-\frac{1}{2}}\mathbf{k}_{sc}(P); \quad (52)$$

$$\mathbf{B}_{cs}(P)=2(2)^{\frac{1}{2}}A_c(Q_{cs})[V_s(P_{cs})]^{-\frac{1}{2}}\left[\frac{V_c(P_{cs})}{V_s(P_{cs})}\right]^{5/4}\left[\frac{V_c(Q_{cs})\rho(Q_{cs})d\sigma(Q_{cs})d\tau(P_{cs})}{V_s(P)\rho(P)d\sigma(P_{cs})d\tau(P)}\right]^{\frac{1}{2}}\cdot[2\sin^2\theta_c-V_c^2V_s^{-2}]_{Q_{cs}}$$
$$\times[2\sin\theta_c\cos\theta_c]_{P_{cs}}\left[\frac{\partial G(\theta_c)}{\partial\theta}\right]_{Q_{cs}}^{-\frac{1}{2}}\left[\frac{\partial G(\theta_c)}{\partial\theta}\right]_{P_{cs}}^{-\frac{1}{2}}\cdot[\kappa_0+(\kappa+\dot\theta_c)]_{Q_{cs}}^{-\frac{1}{2}}\mathbf{n}_{cs}(P); \quad (53)$$

$$\mathbf{B}_{ss}(P)=2(2)^{\frac{1}{2}}A_s(Q_{cs})[V_s(P_{ss})]^{-\frac{1}{2}}\left[\frac{V_c(P_{ss})}{V_s(P_{ss})}\right]^{5/4}\left[\frac{V_c(Q_{ss})}{V_s(Q_{ss})}\right]^{5/4}\cdot\left[\frac{V_s(Q_{ss})\rho(Q_{ss})d\sigma(Q_{ss})d\tau(P_{ss})}{V_s(P)\rho(P)d\sigma(P_{ss})d\tau(P)}\right]^{\frac{1}{2}}$$
$$\cdot[2\sin\theta_c\cos\theta_c]_{P_{ss}}[2\sin\theta_c\cos\theta_c]_{Q_{ss}}\cdot\left[\frac{\partial G(\theta_c)}{\partial\theta}\right]_{Q_{ss}}^{-\frac{1}{2}}\left[\frac{\partial G(\theta_c)}{\partial\theta}\right]_{P_{ss}}^{-\frac{1}{2}}\cdot[\kappa_0+(\kappa+\dot\theta_s)]_{Q_{ss}}^{-\frac{1}{2}}\mathbf{n}_{ss}(P); \quad (54)$$

and the phase $\varphi_{ij}'(P)$, which is independent of ω, is given by $\varphi_{ij}(P)/\omega$.

V. TRANSIENT WAVES

The results obtained in the preceding section apply to time-harmonic waves. We now transform them to apply to transient waves. Suppose that we have a time-harmonic solution $\mathbf{u}(\mathbf{x},\omega)e^{-i\omega t}$ of the elastic equations of motion for a source distribution with harmonic time dependence. Then it is easily shown, by means of Fourier transforms, that the time-dependent solution $\mathbf{U}(\mathbf{x},t)$ for the same source distribution with an arbitrary time dependence $F(t)$ is given by

$$\mathbf{U}(\mathbf{x},t)=\int_{-\infty}^{\infty}\mathbf{u}(\mathbf{x},\omega)f(\omega)e^{-i\omega t}d\omega. \quad (55)$$

Here, $f(\omega)$ is the Fourier transform of $F(t)$ and

$$F(t)=\int_{-\infty}^{\infty}f(\omega)e^{-u\omega t}d\omega. \quad (56)$$

It is possible to use Eq. (55) in its present form to find the waveform for transient waves. However, it is simpler to use another procedure that makes use of the special form of the frequency dependence obtained in Eq. (50). The time-harmonic solution given by Eq. (50) is of the form

$$\mathbf{u}(\mathbf{x},t)=\sqrt{\pi}e^{-i(\pi/4)}\omega^{\frac{1}{2}}\sum_{i,j}\mathbf{B}_{ij}e^{i\omega\varphi_{ij}'}, \quad (57)$$

where the summation consists of four terms $i=c, s$; $j=c, s$. The amplitudes $\mathbf{B}_{ij}$ and phases φ_{ij}', which are independent of frequency, are easily found by comparison of Eq. (57) with (50)–(54). Using Eq. (57) in (55), we obtain

$$\mathbf{U}(\mathbf{x},t)=\sqrt{\pi}e^{-i(\pi/4)}\sum_{i,j}\mathbf{B}_{ij}\int_{-\infty}^{\infty}\omega^{\frac{1}{2}}e^{-i\omega(t-\varphi_{ij}')}f(\omega)d\omega. \quad (58)$$

Now, we differentiate Eq. (56) α times with respect to t and obtain

$$\frac{d^\alpha}{dt^\alpha}F(t)=(-i)^\alpha\int_{-\infty}^{\infty}\omega^\alpha e^{-i\omega t}f(\omega)d\omega. \quad (59)$$

If we let $\alpha=\frac{1}{2}$, then a comparison of Eq. (58) with (59) shows that

$$\mathbf{U}(\mathbf{x},t)=\sqrt{\pi}\sum_{i,j}\mathbf{B}_{ij}(d/dt)^{\frac{1}{2}}F(t-\varphi_{ij}'). \tag{60}$$

Thus, the desired transient waveform for a given input $F(t)$ is given by the one-half-order derivative of $F(t-\varphi_{ij}')$. It is well-known that the one-half-order derivative is given by either of the following expressions[8]:

$$\left(\frac{d}{dt}\right)^{\frac{1}{2}}F(t)=\frac{1}{\sqrt{\pi}}\int_0^t (t-\tau)^{-\frac{1}{2}}\frac{d}{d\tau}F(\tau)d\tau. \tag{61}$$

$$\left(\frac{d}{dt}\right)^{\frac{1}{2}}F(t)=\frac{1}{\sqrt{\pi}}\frac{d}{dt}\int_0^t (t-\tau)^{-\frac{1}{2}}F(\tau)d\tau. \tag{62}$$

Upon substituting Eq. (61) or (62) into (60), we obtain two different expressions for the transient waveform. They are

$$\mathbf{U}(\mathbf{x},t)=\sum_{i,j}\mathbf{B}_{ij}\int_0^{t-\varphi_{ij}'}(t-\varphi_{ij}'-\tau)^{-\frac{1}{2}}\frac{d}{d\tau}F(\tau)d\tau, \tag{63}$$

and

$$\mathbf{U}(\mathbf{x},t)=\sum_{i,j}\mathbf{B}_{ij}\frac{d}{dt^*}\int_0^{t^*}(t^*-\tau)^{-\frac{1}{2}}F(\tau)d\tau, \tag{64}$$

where $t^*=t-\varphi_{ij}'$ is the delayed time. The above equations yield the transient response for an arbitrary pulse $F(t)$ and are the main results of this section.

As a simple example, suppose that $F(t)$ is the unit step function:

$$F(t)=\begin{cases}0, & t<0\\ 1, & t\geq 0\end{cases}. \tag{65}$$

Then, $dF(t)/dt=\delta(t)$, and Eq. (63) yields immediately

$$\mathbf{U}(\mathbf{x},t)=\sum_{i,j}\mathbf{B}_{ij}(t-\varphi_{ij}')^{-\frac{1}{2}}. \tag{66}$$

Other pulses may be treated in a similar manner.

ACKNOWLEDGMENTS

The research reported in this paper was supported by the American Petroleum Institute (Project 61) and the American Chemical Society (Petroleum Research Fund grant in aid 436-A).

[8] R. Courant, *Differential and Integral Calculus* (Interscience Publishers, Inc., New York, 1936), Vol. 2, pp. 339–341.

AFCRC-TN-58-589
ASTIA Document No. AD 207520

NEW YORK UNIVERSITY

Institute of Mathematical Sciences

Division of Electromagnetic Research

Research Report No. EM-119

HOW DARK IS THE SHADOW OF A ROUND-ENDED SCREEN?

Joseph B. Keller

Joseph B. Keller
Joseph B. Keller

Sidney Borowitz
Sidney Borowitz
Acting Director

Dr. Werner W. Gerbes
Dr. Werner Gerbes
Contract Monitor

The research reported in this document has been sponsored by the Electronics Research Directorate of the Air Force Cambridge Research Center, Air Research and Development Command. The publication of this report does not necessarily constitute approval by the Air Force of the findings or conclusions contained herein.

Contract No. AF 19(604)1717

October, 1958

Abstract

The amplitudes of the fields in the shadows of two screens with rounded ends are determined as functions of b/λ. Here b denotes the radius of curvature of the end of the screen and λ denotes the wavelength of the normally incident plane wave. For one of the screens the amplitude decreases monotonically as b/λ increases when the field vanishes on the screen. But if the normal derivative vanishes the amplitude first oscillates as b/λ increases and then finally decreases. For the other screen - the more realistic one - the amplitude probably decreases monotonically in both cases as b/λ increases.

The calculations are based on the geometrical theory of diffraction. For the second screen a calculation from the exact series solution is also made. The close agreement between the two results is a confirmation of the geometrical diffraction theory.

Table of Contents

1. Introduction

Because of diffraction, there is always some illumination in a shadow. The intensity of this illumination depends upon the wavelength λ of the incident radiation and upon the size and shape of the shadow-forming object. In order to examine this dependence, we have considered the shadows cast by two semi-infinite screens with rounded tips of radius b upon either of which a plane wave of unit amplitude is normally incident. (See Figures 1 and 2). We find that in either case, at a large distance r from the tip the diffracted field u in the shadow has the form

$$(1) \qquad u = \frac{e^{i(kr+\frac{\pi}{4})}}{\sqrt{kr}} \; f(\theta,kb) \quad .$$

In (1) $k = 2\pi/\lambda$ is the propagation constant and $f(\theta,kb)$ is the diffraction pattern or complex amplitude of the cylindrical wave scattered from the tip in the direction θ. The angle θ is measured from the shadow boundary.

In Figure 3, the amplitude $|f(\theta,kb)|$ is shown as a function of kb for $\theta = \pi/4$ for the screen in Figure 2. The encircled points show that the shadow cast by this screen becomes darker as kb increases when $u = 0$ on the screen. Thus, the shadow becomes darker as the radius of the tip increases as compared to the wavelength. The factor $k^{-1/2}$ in the denominator of (1) also makes the shadow become darker as the wavelength decreases, independently of the radius of the tip. On the other hand, if $\partial u/\partial n = 0$ then $|f(\frac{\pi}{4},kb)|$ oscillates as kb increases. Ultimately it ceases oscillating and decays monotonically as (6) shows. This oscillation is due to interference between the two diffracted rays which occur in this case. Although the same two rays occur when $u = 0$ on the screen, one of them decays so rapidly that it produces negligible interference and no oscillation.

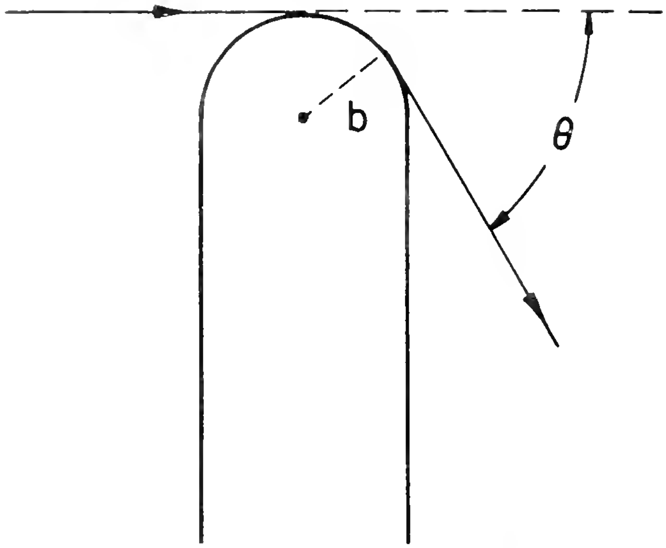


Figure 1 Cross-section of a screen of width 2b with a rounded tip of radius b. A plane wave is normally incident upon it from the left. The dotted line is the shadow boundary. An incident ray which grazes the tip is shown together with one of the diffracted rays which it produces in the shadow. The angle between the diffracted ray and the shadow boundary is θ.

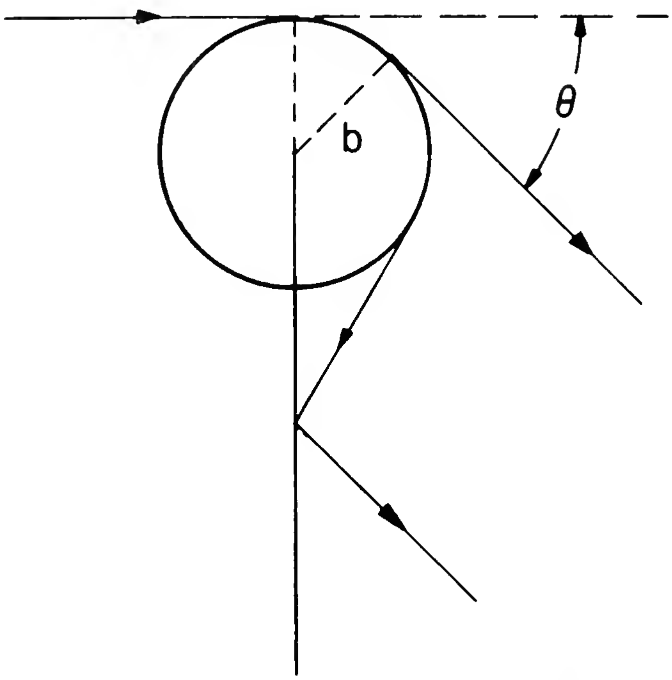


Figure 2 Cross-section of an infinitely thin screen with a cylindrical tip of radius b. A plane wave is normally incident upon it from the left. The dotted line is the shadow boundary. An incident ray which grazes the tip is shown together with the two diffracted rays which it produces making the angle θ with the shadow boundary.

An important feature of Figure 3 is the close agreement between the predictions of the geometrical theory of diffraction and the exact solutions of the boundary value problems for the field.

In Figure 4 the amplitude $|f(\theta,kb)|$ is shown as a function of kb for $\theta = \pi/4$ for the screen of Figure 1. From the physical significance of the approximations on which this figure is based, it is likely that only the decreasing parts of the curves are correct. For this screen the amplitude probably decreases monotonically for all values of kb.

2. Analysis

If the radius of the tip is zero either screen is a half-plane. In this case Sommerfeld's well-known expression for u is of the form (1) with f given by

$$(2) \qquad f(\theta,0) = \frac{1}{2\sqrt{2\pi}} \left[\sec \frac{1}{2}(\theta+\pi) \pm \csc \frac{1}{2}(\theta+\pi) \right] .$$

The upper sign in (2) applies to a screen on which $u = 0$, while the lower sign applies to one on which $\partial u/\partial n = 0$. At the other extreme, when kb is large, the field in the shadow can be determined by means of the geometrical theory of diffraction [1]. In Figures 1 and 2 are shown the one or two diffracted rays which go in a given direction. By the methods and formulas in [1] we find that far from the tip the field associated with the ray in Figure 1 is given by

$$(3) \qquad u_1 = \frac{e^{i(kr+\frac{\pi}{4})}}{\sqrt{kr}} \cdot (kb)^{1/3} \frac{C_o}{\sqrt{2}} e^{i\tau_o (kb)^{1/3}\theta} e^{i(kb\theta + \frac{\pi}{12} - \frac{\pi}{4})} .$$

This same expression applies to the upper ray in Figure 2, while for the lower ray we find

$$(4) \qquad u_2 = \mp \frac{e^{i(kr+\frac{\pi}{4})}}{\sqrt{kr}} \cdot (kb)^{1/3} \frac{C_o}{\sqrt{2}} e^{i\tau_o (kb)^{1/3}(\pi-\theta)} e^{i(kb|\pi-\theta| + \frac{\pi}{12} - \frac{\pi}{4})} .$$

The upper sign in (4) applies if $u = 0$ on the screen, while the lower sign applies if $\partial u/\partial n = 0$. In these two cases the constants C_o and τ_o are given by

	$u = 0$	$\frac{\partial u}{\partial n} = 0$
C_o	.910719	1.53228
τ_o	$1.855757\ e^{i\pi/3}$	$.808617\ e^{i\pi/3}$.

The fields $u = u_1$ and $u = u_1 + u_2$ are of the form (1) with f given by the respective formulas:

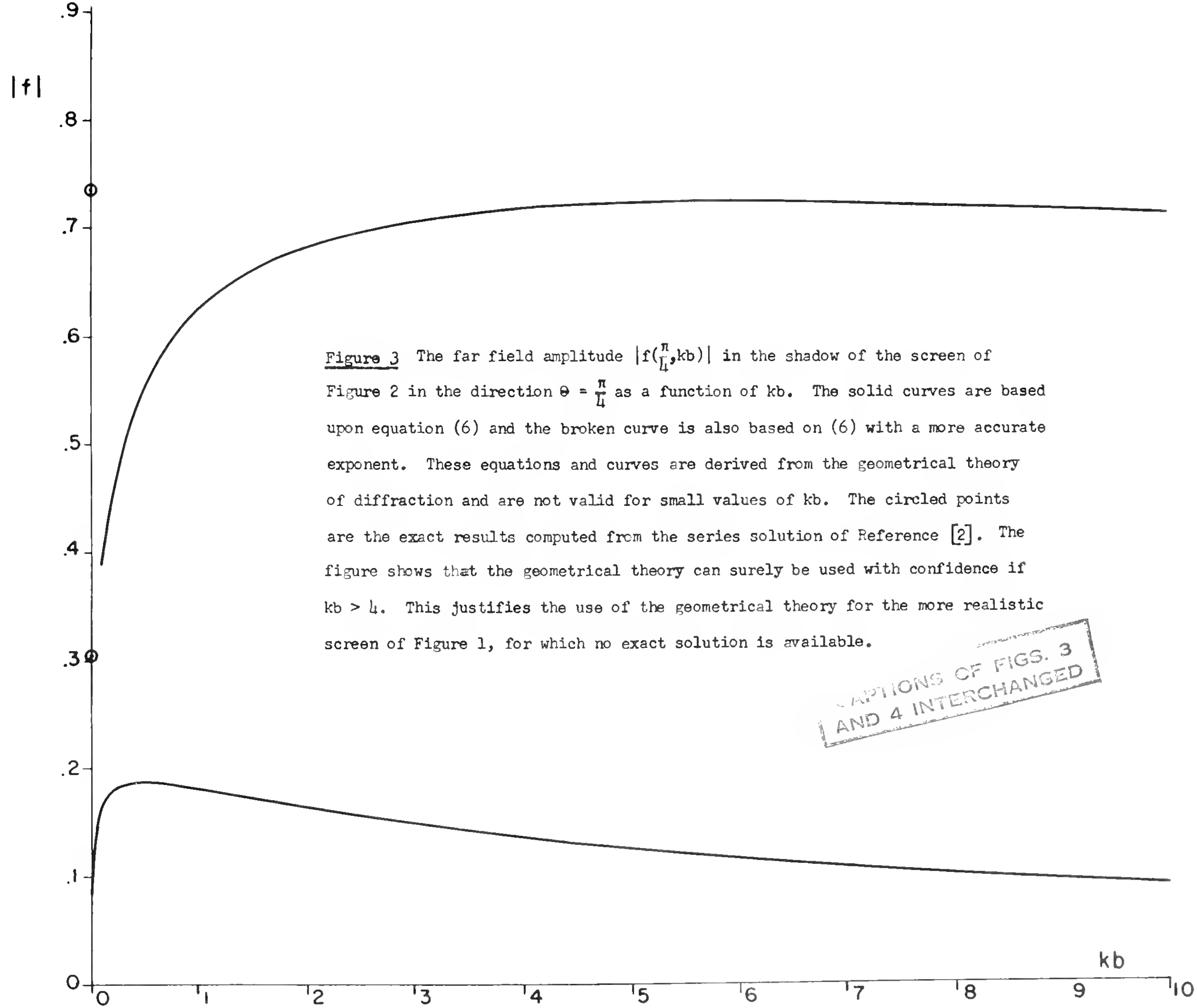
$$(5)\qquad f(\theta,kb) = (kb)^{1/3}\ \frac{C_o}{\sqrt{2}}\ e^{-i\pi/6}\ e^{i\left[kb+\tau_o(kb)^{1/3}\right]\theta}$$

$$(6)\qquad f(\theta,kb) = (kb)^{1/3}\ \frac{C_o}{\sqrt{2}}\ e^{-i\pi/6}\left[e^{i\left[kb+\tau_o(kb)^{1/3}\right]} \mp e^{i\left[kb+\tau_o(kb)^{1/3}\right](\pi-\theta)}\right] .$$

In Figures 3 and 4 the values of $|f(\frac{\pi}{4},kb)|$ are shown based on (2) for $kb = 0$ and on (5) and (6) (solid line) for $kb \geq .1$ for screens on which either $u = 0$ (lower graph) or $\partial u/\partial n = 0$ (upper graph).

For the screen of Figure 2 the field u has been determined exactly for all values of kb by the method of separation of variables [2]. The result for f, which coincides with (2) for $kb = 0$, has been expanded asymptotically for kb large [3]. It then agrees with (6). We have used this solution to calculate $|f|$ in the range $0 < kb \leq 10$. The results are shown as circled points in Figure 3 for both the case in which $u = 0$ (lower graph) and that in which $\partial u/\partial n = 0$ (upper graph) on the screen. The broken line in the upper graph of Figure 3 for the case $\partial u/\partial n = 0$ is based upon (6) with the factor $kb + \tau_o(kb)^{1/3}$ in the exponents replaced by the more accurate expression $kb + \tau_o(kb)^{1/3} - \sigma_o(kb)^{-1/3}$. The constant σ_o has the value [4] $\sigma_o = .145463\ e^{-i\pi/3}$.

The agreement between the circled points based upon the series solution [2] and the simple results (6) of the geometrical theory of diffraction [1] is



<u>Figure 3</u> The far field amplitude $|f(\frac{\pi}{4},kb)|$ in the shadow of the screen of Figure 2 in the direction $\theta = \frac{\pi}{4}$ as a function of kb. The solid curves are based upon equation (6) and the broken curve is also based on (6) with a more accurate exponent. These equations and curves are derived from the geometrical theory of diffraction and are not valid for small values of kb. The circled points are the exact results computed from the series solution of Reference [2]. The figure shows that the geometrical theory can surely be used with confidence if $kb > 4$. This justifies the use of the geometrical theory for the more realistic screen of Figure 1, for which no exact solution is available.

APTIONS OF FIGS. 3
AND 4 INTERCHANGED

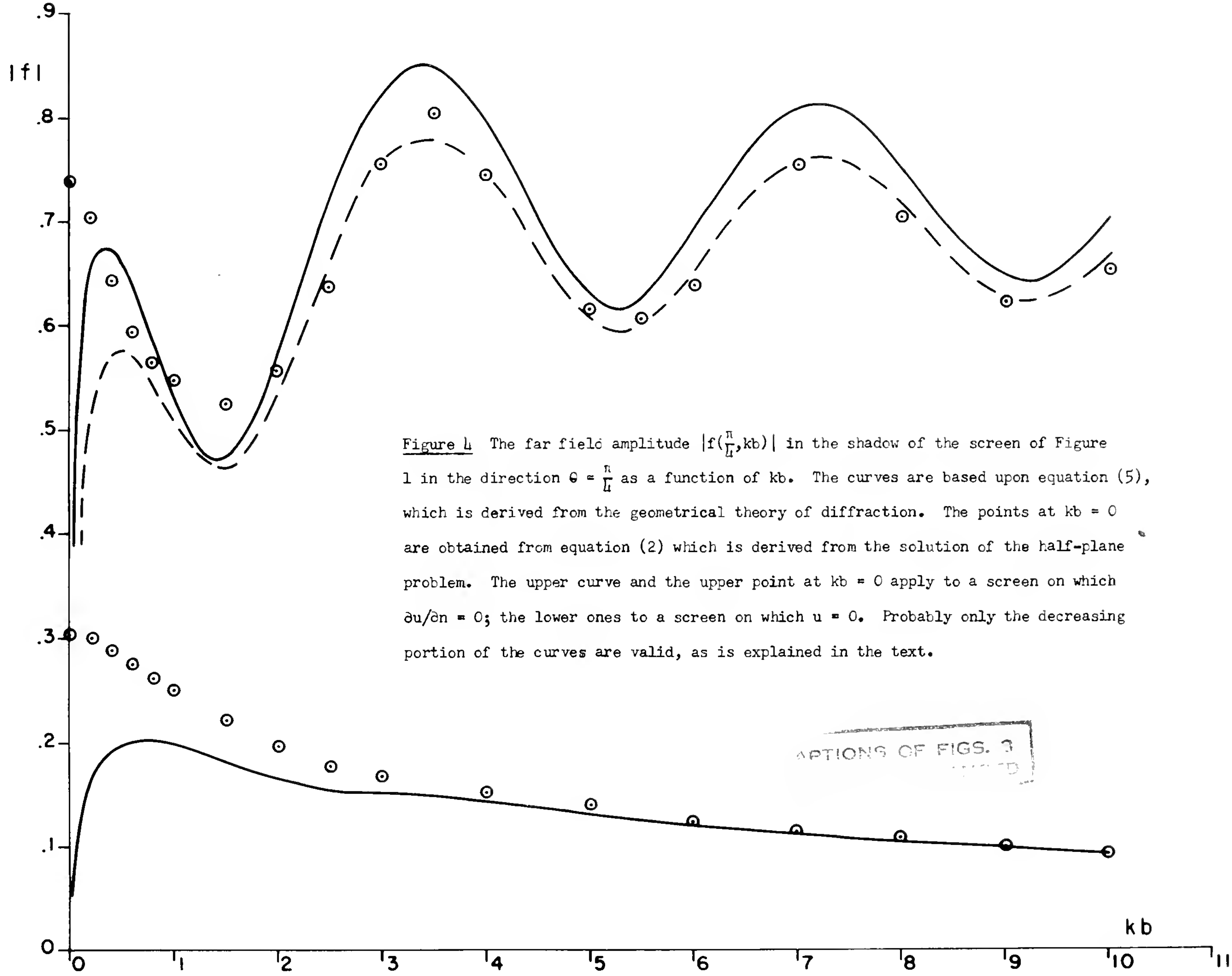


Figure 4 The far field amplitude $|f(\frac{\pi}{4},kb)|$ in the shadow of the screen of Figure 1 in the direction $\theta = \frac{\pi}{4}$ as a function of kb. The curves are based upon equation (5), which is derived from the geometrical theory of diffraction. The points at kb = 0 are obtained from equation (2) which is derived from the solution of the half-plane problem. The upper curve and the upper point at kb = 0 apply to a screen on which $\partial u/\partial n = 0$; the lower ones to a screen on which u = 0. Probably only the decreasing portion of the curves are valid, as is explained in the text.

another confirmation of the geometrical theory. The labor involved in computing the curves from (6) was almost negligible. However, the calculations of the points from the series solution required a great deal of time. Four series of Bessel functions had to be computed for each point and each required about 1.2 kb + 2 terms to yield three decimal accuracy. To use the series for kb > 10 would require tables of Bessel functions of high order as well as many terms. In view of the agreement of the series result with (6), this does not seem to be necessary. Thus, for example, we may conclude from (6) that for kb large $|f|$ ultimately decreases.

Acknowledgment

It is a pleasure to thank Mr. Hans Lehnsen for performing the calculations upon which the figures are based.

References

[1] Keller, J. B. — Diffraction by a convex cylinder; IRE Transactions on Antennas and Propagation, Symposium on Electromagnetic Wave Theory, AP-4, 243 (1956); also, N.Y.U., Inst. of Math. Sci., Div. of EM Research, Report No. EM-94.

[2] Karp, S. N. — Diffraction by a tipped wedge - with applications to blunt edges; N.Y.U., Inst. of Math. Sci., Div. of EM Research, Report No. EM-52.

[3] Magiros, D. and J. B. Keller — Diffraction by a screen with a rounded end; future EM report.

[4] Levy, B. and J. B. Keller — Diffraction by a Smooth Object; N.Y.U., Inst. of Math. Sci., Div. of EM Research, Report No. EM-109.

DISTRIBUTION LIST FOR RESEARCH REPORTS

Contract No. AF 19(604)1717

(ONE copy unless otherwise noted)

Commander
Air Research and Development Command
Andrews Air Force Base
Washington 25, D. C.
Attn: Major E. Wright, RDTCC

Director of Resident Training
3380th Technical Training Group
Keesler Air Force Base, Mississippi
Attn: OA-3011 Course

Director
Air University Library
Maxwell Air Force Base, Alabama

Commander
Air Force Missile Test Center
Patrick Air Force Base, Florida
Attn: MTE - for classified documents
Attn: MU-411, Technical Library - for unclassified documents

Tactical Air Group
Directorate of Research and Development DCS/D
Headquarters, USAF
Washington, D. C.
Attn: Major R. L. Stell

Director, Communications and Electronics
Hq. U. S. Air Force
Washington 25, D. C.
Attn: AFOAC S/E

Commander
Wright Air Development Center
Wright-Patterson Air Force Base, Ohio
Attn: WCLRS-6, Mr. Portune

Wright Air Development Center
Wright-Patterson Air Force Base, Ohio
Flight Research Laboratory
Research Division
Attn: WCRRA

Commander
Wright Air Development Center
Wright-Patterson Air Force Base, Ohio
Attn: N. Draganjac, WCLNQ-4

Commander
Wright Air Development Center
Wright-Patterson Air Force Base, Ohio
Attn: Mr. Paul Springer, WCLRE-5

Commander
Air Technical Intelligence Center
Wright-Patterson Air Force Base, Ohio
Attn: AFCIN-4Bia

Commander
Rome Air Development Center
Griffiss Air Force Base, New York
Attn: RCSSTL-1

Commander
Rome Air Development Center
Griffiss Air Force Base, New York
Attn: Mr. Donald Dakan, RCUE

Commander
Rome Air Development Center (ARDC)
Griffiss Air Force Base, New York
Attn: Dr. John S. Burgess, RCE

Commander
Air Force Missile Development Center
Holloman Air Force Base, New Mexico
Attn: HDOIL, Technical Library

Director
U. S. Army Ordnance
Ballistic Research Laboratories
Aberdeen Proving Ground, Maryland
Attn: Ballistic Measurements Laboratory

Ballistic Research Laboratories
Aberdeen Proving Ground, Maryland
Attn: Technical Information Branch

Director
Evans Signal Laboratory
Belmar, New Jersey
Attn: Mr. O. C. Woodyard

U. S. Army Signal Engineering Labs.
Evans Signal Laboratory
Belmar, New Jersey
Attn: Technical Document Center

Massachusetts Institute of Technology
Signal Corps Liaison Officer
Cambridge 39, Mass.
Attn: A. D. Bedrosian, Room 26-131

Commanding General, SIGFM/EL-PC
U. S. Army Signal Engineering Labs.
Fort Monmouth, New Jersey
Attn: Dr. Horst H. Kedesdy
Deputy Chief, Chem-Physics Branch

Commander
Army Rocket and Guided Missile Agency
Redstone Arsenal, Alabama
Attn: Technical Library, ORDXR-OTL

Commanding General
U. S. Army Signal Engineering Labs.
Fort Monmouth, New Jersey
Attn: SIGFM/EL-AT

Department of the Army
Office of the Chief Signal Officer
Washington 25, D. C.
Attn: SIGRD-7

Office of Chief Signal Officer
Engineering and Technical Division
Washington 25, D. C.
Attn: SIGNET-5

Guided Missile Fuze Library
Diamond Ordnance Fuze Laboratories
Washington 25, D. C.
Attn: R. D. Hatcher, Chief Microwave Development Section

(10) Armed Services Technical Information Agency
Arlington Hall Station
Arlington 12, Virginia

(2) Library
Boulder Laboratories
National Bureau of Standards
Boulder, Colorado

National Bureau of Standards
Department of Commerce
Washington 25, D. C.
Attn: Mr. A. G. McNish

National Bureau of Standards
Department of Commerce
Washington 25, D. C.
Attn: Gustave Shapiro, Chief
Engineering Electronics Section
Electricity and Electronics Div.

(2) Office of Technical Services
Department of Commerce
Washington 25, D. C.
Attn: Technical Reports Section
(Unclassified only)

Director
National Security Agency
Washington 25, D. C.
Attn: R/D (331)

(2) Hq. Air Force Cambridge Research Center
Laurence G. Hanscom Field
Bedford, Mass.
Attn: CROTLR-2 - P. Condon

(5) Hq. Air Force Cambridge Research Center
Laurence G. Hanscom Field
Bedford, Mass.
Attn: CROTLS - J. Armstrong

(5) Hq. Air Force Cambridge Research Center
Laurence G. Hanscom Field
Bedford, Mass.
Attn: CRRD

Director, Avionics Division (AV)
Bureau of Aeronautics
Department of the Navy
Washington 25, D. C.

Chief, Bureau of Ships
Department of the Navy
Washington 25, D. C.
Attn: Mr. E. Johnston, Code 833E

Commander
U. S. Naval Air Missile Test Center
Point Mugu, California
Attn: Code 366

U. S. Naval Ordnance Laboratory
White Oak
Silver Spring 19, Maryland
Attn: The Library

Commander
U. S. Naval Ordnance Test Station
China Lake, California
Attn: Code 753

Librarian
U. S. Naval Postgraduate School
Monterey, California

Air Force Development Field Representative
Naval Research Laboratory
Washington 25, D. C.
Attn: Code 1072

Director
U. S. Naval Research Laboratory
Washington 25, D. C.
Attn: Code 2027

Dr. J. I. Bohnert, Code 5210
U. S. Naval Research Laboratory
Washington 25, D. C. (Unclassified only)
Classified to be sent to:
Director
U. S. Naval Research Laboratory
Attn: Code 5200
Washington 25, D. C.

Commanding Officer and Director
U. S. Navy Underwater Sound Laboratory
Fort Trumbull, New London, Connecticut

Chief of Naval Research
Department of the Navy
Washington 25, D. C.
Attn: Code 427

Commanding Officer and Director
U. S. Navy Electronics Laboratory (Library)
San Diego 52, California

Chief, Bureau of Ordnance
Department of the Navy
Washington 25, D. C.
Attn: Code Ad3

Chief, Bureau of Ordnance
Department of the Navy
Surface Guided Missile Branch
Washington 25, D. C.
Attn: Code ReS1-e

Chief, Bureau of Ordnance
Department of the Navy
Washington 25, D. C.
Attn: Fire Control Branch (ReS4)

Department of the Navy
Bureau of Aeronautics
Technical Data Division, Code 4106
Washington 25, D. C.

Chief, Bureau of Ships
Department of the Navy
Washington 25, D. C.
Attn: Code 817B

Commanding Officer
U. S. Naval Air Development Center
Johnsville, Pennsylvania
Attn: NADC Library

Commander
U. S. Naval Air Test Center
Patuxent River, Maryland
Attn: ET-315, Antenna Branch

Director
Naval Ordnance Laboratory
Corona, California

Commanding Officer
U. S. Naval Ordnance Laboratory
Corona, California
Attn: Mr. W. Horenstein, Division 72

Airborne Instruments Laboratory, Inc.
160 Old Country Road
Mineola, New York
Attn: Dr. E. G. Fubini, Director
Research and Engineering Division

Aircom, Inc.
354 Main Street
Winthrop, Mass.

American Machine and Foundry Company
Electronics Division
1085 Commonwealth Avenue
Boston 15, Mass.
Attn: Mrs. Rita Moravcsik, Librarian

Andrew Alford, Consulting Engineers
299 Atlantic Avenue
Boston 10, Mass.

Avion Division
ACF Industries, Inc.
800 No. Pitt Street
Alexandria, Virginia
Attn: Library

Battelle Memorial Institute
505 King Avenue
Attn: Wayne E. Rife, Project Leader
Electrical Engineering Division
Columbus 1, Ohio

Bell Aircraft Corporation
Post Office Box One
Buffalo 5, New York
Attn: Eunice P. Hazelton, Librarian

Bell Telephone Laboratories, Inc.
Whippany Laboratory
Whippany, New Jersey
Attn: Technical Information Library

Pacific Division
Bendix Aviation Corporation
11600 Sherman Way
North Hollywood, California
Engineering Library
Attn: Peggie Robinson, Librarian

Bendix Radio Division
Bendix Aviation Corp.
E. Joppa Road
Towson 4, Maryland
Attn: Dr. D. M. Allison, Jr.
Director Engineering and Research

Boeing Airplane Company
Pilotless Aircraft Division
P.O. Box 3707
Seattle 24, Washington
Attn: R.R. Barber, Library Supervisor

Boeing Airplane Company
Wichita Division Engineering Library
Wichita 1, Kansas
Attn: Kenneth C. Knight, Librarian

Boeing Airplane Company
Seattle Division
Seattle 14, Washington
Attn: E.T. Allen, Library Supervisor

Bjorksten Research Labs, Inc.
P. O. Box 265
Madison, Wisconsin
Attn: Mrs. Fern B. Korsgard

Convair, A Division of General Dynamics Corp.
Fort Worth, Texas
Attn: K.G. Brown, Division Research Librarian

Convair, A Division of General Dynamics Corp.
San Diego 12, California
Attn: Mrs. Dora B. Burke,
Engineering Librarian

Cornell Aeronautical Laboratory, Inc.
4455 Genesee Street
Buffalo 21, New York
Attn: Librarian

Dalmo Victor Company
A Division of Textron, Inc.
1515 Industrial Way
Belmont, California
Attn: Mary Ellen Addems,
Technical Librarian

Dorne and Margolin, Inc.
29 New York Avenue
Westbury, Long Island, N. Y.

Douglas Aircraft Company, Inc.
P.O. Box 200
Long Beach 1, California
Attn: Engineering Library (C-250)

Douglas Aircraft Co., Inc.
827 Lapham Street
El Segundo, California
Attn: Engineering Library

Douglas Aircraft Company, Inc.
3000 Ocean Park Boulevard
Santa Monica, California
Attn: P.T. Cline
Eq. Sec. Reference Files,
Eq. Eng. A250

Douglas Aircraft Company, Inc.
2000 North Memorial Drive
Tulsa, Oklahoma
Attn: Engineering Library, D-250

Electronics Communication, Inc.
1830 York Road
Timonium, Maryland

Emerson and Cuming, Inc.
869 Washington Street
Canton, Mass.
Attn: Mr. W. Cuming

Emerson Electric Mfg. Co.
8100 West Florissant Avenue
St. Louis 21, Missouri
Attn: Mr. E.R. Breslin, Librarian

Sylvania Elec. Prod. Inc.
Electronic Defense Laboratory
P.O. Box 205 - (Uncl)
Mountain View, California
Attn: Library

Fairchild Aircraft Division
Fairchild Eng. and Airplane Corp.
Hagerstown, Maryland
Attn: Library

Farnsworth Electronics Company
3700 East Pontiac Street
Fort Wayne 1, Indiana
Attn: Technical Library

Federal Telecommunication Labs.
500 Washington Avenue
Nutley 10, New Jersey
Attn: Technical Library

The Gabriel Electronics
Division of the Gabriel Company
135 Crescent Road
Needham Heights 94, Mass.
Attn: Mr. Steven Galagan

General Electric Advanced Electronics Center
Cornell University
Ithaca, New York
Attn: J. B. Travis

General Electric Company
Electronics Park
Syracuse, New York
Attn: Documents Library, B. Fletcher
Building 3-143A

General Precision Laboratory, Inc.
63 Bedford Road
Pleasantville, New York
Attn: Mrs. Mary G. Herbst, Librarian

Goodyear Aircraft Corp.
1210 Massillon Road
Akron 15, Ohio
Attn: Library D/120 Plant A

Granger Associates
Electronic Systems
966 Commercial Street
Palo Alto, California
Attn: John V. N. Granger, President

Grumman Aircraft Engineering Corporation
Bethpage, Long Island, N. Y.
Attn: Mrs. A. M. Gray, Librarian
Engineering Library, Plant No. 5

The Hallicrafters Company
4401 West 5th Avenue
Chicago 24, Illinois
Attn: LaVerne LaGioia, Librarian

Hoffman Laboratories, Inc.
3761 South Hill Street
Los Angeles 7, California
Attn: Engineering Library

Hughes Aircraft Company
Antenna Department
Microwave Laboratory
Building 12, Room 2617
Culver City, California
Attn: M. D. Adcock

Hughes Aircraft Company
Florence and Teale Streets
Culver City, California
Attn: Dr. L.C. Van Atta, Associate Director
Research Labs.

Hycon Eastern, Inc.
75 Cambridge Parkway
Cambridge, Mass.
Attn: Mrs. Lois Seulowitz
Technical Librarian

International Business Machines Corp.
Military Products Division
590 Madison Avenue
New York 33, New York
Attn: Mr. C.F. McElwain, General Manager

International Business Machines Corp.
Military Products Division
Owego, New York
Attn: Mr. D. I. Marr, Librarian
Department 459

International Resistance Company
401 N. Broad Street
Philadelphia 8, Pa.
Attn: Research Library

Jansky and Bailey, Inc.
1339 Wisconsin Avenue, N. W.
Washington 7, D. C.
Attn: Mr. Delmer C. Ports

Dr. Henry Jasik, Consulting Engineer
298 Shames Drive
Brush Hollow Industrial Park
Westbury, New York

Electromagnetic Research Corporation
711 14th Street, N. W.
Washington 5, D. C.

Lockheed Aircraft Corporation
2555 N. Hollywood Way
California Division Engineering Library
Department 72-75, Plant A-1, Bldg. 63-1
Burbank, California
Attn: N. C. Harnois

The Martin Company
P. O. Box 179
Denver 1, Colorado
Attn: Mr. Jack McCormick

The Glenn L. Martin Company
Baltimore 3, Maryland
Attn: Engineering Library
Antenna Design Group

Maryland Electronic Manufacturing Corp.
5009 Calvert Road
College Park, Maryland
Attn: Mr. H. Warren Cooper

Mathematical Reviews
190 Hope Street
Providence 6, Rhode Island

The W. L. Maxson Corporation
460 West 34th Street
New York, N. Y.
Attn: Miss Dorothy Clark

McDonnell Aircraft Corporation
Lambert Saint-Louis Municipal Airport
Box 516, St. Louis 3, Missouri
Attn: R. D. Detrich, Engineering Library

McMillan Laboratory, Inc.
Brownville Avenue
Ipswich, Massachusetts
Attn: Security Officer, Document Room

Melpar, Inc. (2)
3000 Arlington Boulevard
Falls Church, Virginia
Attn: Engineering Technical Library

Microwave Development Laboratory
90 Broad Street
Babson Park 57, Massachusetts
Attn: N. Tucker, General Manager

Microwave Radiation Company Inc.
19223 South Hamilton Street
Gardena, California
Attn: Mr. Morris J. Ehrlich, President

Chance Vought Aircraft, Inc.
9314 West Jefferson Street
Dallas, Texas
Attn: Mr. H. S. White, Librarian

Northrop Aircraft, Inc.
Hawthorne, California
Attn: Mr. E. A. Freitas, Library Dept 3145
1001 E. Broadway

Remington Rand Univ. - Division of Sperry Rand Corporation
1900 West Allegheny Avenue
Philadelphia 29, Pennsylvania
Attn: Mr. John F. McCarthy
R and D Sales and Contracts

North American Aviation, Inc.
12214 Lakewood Boulevard
Downey, California
Attn: Engineering Library 495-115

North American Aviation, Inc.
Los Angeles International Airport
Los Angeles 45, California
Attn: Engineering Technical File

Page Communications Engineers, Inc.
710 Fourteenth Street, Northwest
Washington 5, D. C.
Attn: Librarian

Philco Corporation Research Division
Branch Library
4700 Wissachickon Avenue
Philadelphia 44, Pa.
Attn: Mrs. Dorothy S. Collins

Pickard and Burns, Inc.
240 Highland Avenue
Needham 94, Mass.
Attn: Dr. J. T. DeBettencourt

Polytechnic Research and Development Company, Inc.
202 Tillary Street
Brooklyn 1, New York
Attn: Technical Library

Radiation Engineering Laboratory
Main Street
Maynard, Mass.
Attn: Dr. John Ruze

Radiation, Inc.
P. O. Drawer 37
Melbourne, Florida
Attn: Technical Library, Mr. M.L. Cox

Radio Corp. of America
RCA Laboratories
Rocky Point, New York
Attn: P. S. Carter, Lab. Library

RCA Laboratories
David Sarnoff Research Center
Princeton, New Jersey
Attn: Miss Fern Cloak, Librarian
Research Library

Radio Corporation of America
Defense Electronic Products
Building 10, Floor 7
Camden 2, New Jersey
Attn: Mr. Harold J. Schrader
Staff Engineer, Organization of Chief Technical Administrator

The Ramo-Wooldridge Corporation
P.O. Box 45453 Airport Station
Los Angeles 45, California
Attn: Margaret C. Whitnah,
Chief Librarian

Roover Microwave Co.
9592 Baltimore Avenue
College Park, Maryland

Director, USAF Project RAND
Via: Air Force Liaison Office
The Rand Corporation
1700 Main Street
Santa Monica, California

Rantec Corporation
Calabasas, California
Attn: Grace Keener, Office Manager

Raytheon Manufacturing Company
Missile Systems Division
Bedford, Mass.
Attn: Mr. Irving Goldstein

Raytheon Manufacturing Company
Wayland Laboratory, State Road
Wayland, Mass.
Attn: Mr. Robert Borts

Raytheon Manufacturing Company
Wayland Laboratory
Wayland, Mass.
Attn: Miss Alice G. Anderson,
Librarian

Republic Aviation Corporation
Farmingdale, Long Island, N. Y.
Attn: Engineering Library

Thru: Air Force Plant Representative
Republic Aviation Corp.
Farmingdale, Long Island, N.Y.

Rheem Manufacturing Company
9236 East Hall Road
Downey, California
Attn: J. C. Joerger

Trans-Tech, Inc.
P. O. Box 346
Frederick, Maryland

Ryan Aeronautical Company
Lindbergh Field
San Diego 12, California
Attn: Library - unclassified

Sage Laboratories
159 Linden Street
Wellesley 81, Mass.

Sanders Associates
95 Canal Street
Nashua, New Hampshire
Attn: N. R. Wild, Library

Sandia Corporation, Sandia Base
P.O. Box 5800, Albuquerque, New Mexico
Attn: Classified Document Division

Sperry Gyroscope Company
Great Neck, Long Island, New York
Attn: Florence W. Turnbull, Engr. Librarian

Stanford Research Institute
Menlo Park, California
Attn: Library, Engineering Division

Sylvania Electric Products, Inc.
100 First Avenue
Waltham 54, Mass.
Attn: Charles A. Thornhill, Report Librarian
Waltham Laboratories Library

Systems Laboratories Corporation
14852 Ventura Boulevard
Sherman Oaks, California
Attn: Donald L. Margerum

TRG, Inc.
17 Union Square West
New York 3, N. Y.
Attn: M. L. Henderson, Librarian

A. S. Thomas, Inc.
161 Devonshire Street
Boston 10, Mass.
Attn: A. S. Thomas, President

Bell Telephone Laboratories
Murray Hill
New Jersey

Chu Associates
P. O. Box 387
Whitcomb Avenue
Littleton, Mass.

Microwave Associates, Inc.
Burlington, Mass.

Raytheon Manufacturing Company
Missile Division
Hartwell Road
Bedford, Mass.

Radio Corporation of America
Aviation Systems Laboratory
225 Crescent Street
Waltham, Mass.

Lockheed Aircraft Corporation
Missile Systems Division Research Library
Box 504, Sunnyvale, California
Attn: Miss Eva Lou Robertson,
Chief Librarian

The Rand Corporation
1700 Main Street
Santa Monica, California
Attn: Dr. W. C. Hoffman

Commander
AF Office of Scientific Research
Air Research and Development Command
14th Street and Constitution Avenue
Washington, D. C.
Attn: Mr. Otting, SRY

Westinghouse Electric Corp.
Electronics Division
Friendship Int'l Airport Box 746
Baltimore 3, Maryland
Attn: Engineering Library

Wheeler Laboratories, Inc.
122 Cutter Mill Road
Great Neck, New York
Attn: Mr. Harold A. Wheeler

Zenith Plastics Co.
Box 91
Gardena, California
Attn: Mr. S. S. Oleesky

Library Geophysical Institute
of the University of Alaska
College
Alaska

University of California
Berkeley 4, California
Attn: Dr. Samuel Silver,
Prof. Engineering Science
Division of Elec. Eng. Electronics
Research Lab.

University of California
Electronics Research Lab.
332 Cory Hall
Berkeley 4, California
Attn: J. R. Whinnery

California Institute of Technology
Jet Propulsion Laboratory
4800 Oak Grove Drive
Pasadena, California
Attn: Mr. I. E. Newlan

California Institute of Technology
1201 E. California Street
Pasadena, California
Attn: Dr. C. Papas

Carnegie Institute of Technology,
Schenley Park
Pittsburgh 13, Pennsylvania
Attn: Prof. A. E. Heins

Cornell University
School of Electrical Engineering
Ithaca, New York
Attn: Prof. G. C. Dalman

University of Florida
Department of Electrical Engineering
Gainesville, Florida
Attn: Prof. M. H. Latour, Library

Library
Georgia Institute of Technology
Engineering Experiment Station
Atlanta, Georgia
Attn: Mrs. J.H. Crosland, Librarian

Harvard University
Technical Reports Collection
Gordon McKay Library, 303A Pierce Hall
Oxford Street, Cambridge 38, Mass.
Attn: Mrs. E.L. Hufschmidt, Librarian

Harvard College Observatory
60 Garden Street
Cambridge 39, Mass.
Attn: Dr. Fred L. Whipple

University of Illinois
Documents Division Library
Urbana, Illinois

University of Illinois
College of Engineering
Urbana, Illinois
Attn: Dr. P. E. Moyes, Department of
Electrical Engineering

The Johns Hopkins University
Homewood Campus
Department of Physics
Baltimore 18, Maryland
Attn: Dr. Donald E. Kerr

Sandia Corporation
Attn: Organization 1423
Sandia Base
Albuquerque, New Mexico

Applied Physics Laboratory
The Johns Hopkins University
8621 Georgia Avenue
Silver Spring, Maryland
Attn: Mr. George L. Seielstad

Massachusetts Institute of Technology
Research Laboratory of Electronics
Room 20B-221
Cambridge 39, Massachusetts
Attn: John H. Hewitt

Massachusetts Institute of Technology
Lincoln Laboratory
P. O. Box 73
Lexington 73, Mass.
Attn: Document Room A-229

University of Michigan
Electronic Defense Group
Engineering Research Institute
Ann Arbor, Michigan
Attn: J. A. Boyd, Supervisor

University of Michigan
Engineering Research Institute
Radiation Laboratory
Attn: Prof. K. M. Siegel
912 N. Main St.,
Ann Arbor, Michigan

University of Michigan
Engineering Research Institute
Willow Run Laboratories
Willow Run Airport
Ypsilanti, Michigan
Attn: Librarian

University of Minnesota
Minneapolis 14, Minnesota
Attn: Mr. Robert H. Stumm, Library

Northwestern University
Microwave Laboratories
Evanston, Illinois
Attn: R. E. Beam

Ohio State University Research Found.
Ohio State University
Columbus 10, Ohio
Attn: Dr. T.E. Tice
Dept. of Elec. Engineering

The University of Oklahoma
Research Institute
Norman, Oklahoma
Attn: Prof. C. L. Farrar, Chairman
Electrical Engineering

Polytechnic Institute of Brooklyn
Microwave Research Institute
55 Johnson Street
Brooklyn, New York
Attn: Dr. Arthur A. Oliner

Polytechnic Institute of Brooklyn
Microwave Research Institute
55 Johnson Street
Brooklyn, New York
Attn: Mr. A. E. Laemmel

Syracuse University Research Institute
Collendale Campus
Syracuse 10, New York
Attn: Dr. C. S. Grove, Jr.
Director of Engineering Research

The University of Texas
Elec. Engineering Research Laboratory
P. O. Box 8026, University Station
Austin 12, Texas
Attn: Mr. John R. Gerhardt
Assistant Director

The University of Texas
Defense Research Laboratory
Austin, Texas
Attn: Claude W. Horton, Physics Library

University of Toronto
Department of Electrical Engineering
Toronto, Canada
Attn: Prof. G. Sinclair

Lowell Technological Institute
Research Foundation
P. O. Box 709, Lowell, Mass.
Attn: Dr. Charles R. Mingins

University of Washington
Department of Electrical Engineering
Seattle 5, Washington
Attn: G. Held, Associate Professor

Stanford University
Stanford, California
Attn: Dr. Chodorow
Microwave Laboratory

Physical Science Laboratory
New Mexico College of Agriculture
and Mechanic Arts
State College, New Mexico
Attn: Mr. H. W. Haas

Brown University
Department of Electrical Engineering
Providence, Rhode Island
Attn: Dr. C. M. Angulo

Case Institute of Technology
Cleveland, Ohio
Attn: Prof. S. Seeley

Columbia University
Department of Electrical Engineering
Morningside Heights
New York, N. Y.
Attn: Dr. Schlesinger

McGill University
Montreal, Canada
Attn: Prof. G. A. Woonton
Director, The Eaton Electronics
Research Lab.

Purdue University
Department of Electrical Engineering
Lafayette, Indiana
Attn: Dr. Schultz

The Pennsylvania State University
Department of Electrical Engineering
University Park, Pennsylvania

University of Pennsylvania
Institute of Cooperative Research
3400 Walnut Street
Philadelphia, Pennsylvania
Attn: Dept. of Electrical Engineering

University of Tennessee
Ferris Hall
W. Cumberland Avenue
Knoxville 16, Tennessee

University of Wisconsin
Department of Electrical Engineering
Madison, Wisconsin
Attn: Dr. Scheibe

University of Seattle
Department of Electrical Engineering
Seattle, Washington
Attn: Dr. D. K. Reynolds

Wayne University
Detroit, Michigan
Attn: Prof. A. F. Stevenson

Electronics Research Laboratory
Illinois Institute of Technology
3300 So. Federal Street
Chicago 16, Illinois
Attn: Dr. Lester C. Peach
Research Engineer

Advisory Group on Electronic Parts
Room 103
Moore School Building
200 South 33rd Street
Philadelphia 4, Pennsylvania

Ionosphere Research Laboratory
Pennsylvania State College
State College, Pennsylvania
ATTN: Professor A. H. Waynick, Director

Institute of Mathematical Sciences
25 Waverly Place
New York 3, New York
ATTN: Librarian

Electronics Division
Rand Corporation
1700 Main Street
Santa Monica, California
ATTN: Dr. Robert Kalaba

National Bureau of Standards
Washington, D. C.
ATTN: Dr. W. K. Saunders

Applied Mathematics and Statistics Lab.
Stanford University
Stanford, California
ATTN: Dr. Albert H. Bowker

Department of Physics and Astronomy
Michigan State College
East Lansing, Michigan
ATTN: Dr. A. Leitner

University of Tennessee
Knoxville, Tennessee
ATTN: Dr. Fred A. Ficken

Lebanon Valley College
Annville, Pennsylvania
ATTN: Professor B.H. Bissinger

General Atomic
P. O. Box 608
San Diego 12, California
Attn: Dr. Edward Gerjuoy

Department of Physics
Amherst College
Amherst, Mass.
ATTN: Dr. Arnold Arons

California Institute of Technology
1201 E. California Street
Pasadena, California
ATTN: Dr. A. Erdelyi

Mathematics Department
Stanford University
Stanford, California
ATTN: Dr. Harold Levine

University of Minnesota
Minneapolis 14, Minnesota
ATTN: Professor Paul C. Rosenbloom

Department of Mathematics
Stanford University
Stanford, California
ATTN: Professor Bernard Epstein

Applied Physics Laboratory
The Johns Hopkins University
8621 Georgia Avenue
Silver Spring, Maryland
ATTN: Dr. B. S. Gourary

(2) Exchange and Gift Division
The Library of Congress
Washington 25, D. C.

Electrical Engineering Department
Massachusetts Institute of Technology
Cambridge 39, Mass.
ATTN: Dr. L. J. Chu

Nuclear Development Associates, Inc.
5 New Street
White Plains, New York
ATTN: Library

California Institute of Technology
Electrical Engineering
Pasadena, California
ATTN: Dr. Zohrab A. Kaprielian

Dr. Rodman Doll
311 W. Cross Street
Ypsilanti, Michigan

California Inst. of Technology
Pasadene, California
ATTN: Mr. Calvin Wilcox

(3) Mr. Robert Brockhurst
Woods Hole Oceanographic Institute
Woods Hole, Mass.

National Bureau of Standards
Boulder, Colorado
ATTN: Dr. R. Gallet

Dr. Solomon L. Schwebel
3689 Louis Road
Palo Alto, California

University of Minnesota
The University of Library
Minneapolis 14, Minnesota
ATTN: Exchange Division

Department of Mathematics
University of California
Berkeley, California
ATTN: Professor Bernard Friedman

Lincoln Laboratory
Massachusetts Institute of Technology
P. O. Box 73
Lexington 73, Massachusetts
ATTN: Dr. Shou Chin Wang, Room C-351

Melpar, Inc.,
3000 Arlington Boulevard
Falls Church, Virginia
ATTN: Mr. K. S. Kelleher, Section Head

Hq. Air Force Cambridge Research Center
Laurence G. Hanscom Field
Bedford, Mass.
ATTN: Mr. Francis J. Zucker, CRRD

Hq. Air Force Cambridge Research Center
Laurence G. Hanscom Field
Bedford, Mass.
ATTN: Dr. Philip Newman, CRRK

Mr. N. C. Gerson
Trepelo Road
South Lincoln, Mass.

Dr. Richard B. Barrar
Systems Development Corp.
2400 Colorado Avenue
Santa Monica, California

Columbia University Hudson Laboratories
P.O. Box 239
145 Palisade Street, Dobbs Ferry, N. Y.
ATTN: Dr. N. W. Johnson

Institute of Fluid Dynamics
and Applied Mathematics
University of Maryland
College Park, Maryland
ATTN: Dr. Elliott Montroll

Department of Electrical Engineering
Washington University
Saint Louis 5, Mo.
ATTN: Professor J. Van Bladel

Department of the Navy
Office of Naval Research Branch Office
1030 E. Green Street
Pasadena 1, California

Brandeis University
Waltham, Mass.
ATTN: Library

General Electric Company
Mictowave Laboratory
Electronics Division
Stanford Industrial Park
Palo Alto, California
ATTN: Library

Smyth Research Associates
3555 Aero Court
San Diego 3, California
ATTN: Dr. John B. Smyth

Electrical Engineering
California Institute of Technology
Pasadena, California
ATTN: Dr. Georges G. Weill

Naval Research Laboratory
Washington 25, D. C.
ATTN: Henry J. Passerini, Code 5278A

Dr. George Kear
5 Culver Court
Orinda, California

Brooklyn Polytechnic
85 Livingston Street
Brooklyn, New York
ATTN: Dr. Nathan Marcuvitz

Department of Electrical Engineering
Brooklyn Polytechnic
85 Livingston Street
Brooklyn, New York
ATTN: Dr. Jerry Shmoys

Department of Mathematics
University of New Mexico
Albuquerque, New Mexico
ATTN: Dr. I. Kolodner

Mathematics Department
Polytechnic Institute of Brooklyn
Johnson and Jay Street
Brooklyn, New York
ATTN: Dr. Harry Hochstadt

Ballistics Research Laboratory
Aberdeen Proving Grounds
Aberdeen, Maryland
ATTN: Dr. Pullen Keats

Dr. Lester Kraus
4935 Whitehaven Way
San Diego, California

University of Minnesota
Institute of Technology
Minneapolis, Minnesota
Attn: Dean Athelston Spilhaus

Ohio State University
Columbus, Ohio
Attn: Prof. C. T. Tai
Department of Electrical Eng.

Naval Research Laboratories
Washington 25, D. C.
Attn: W. S. Ament, Code 5271

Naval Research Laboratory
Washington 25, D. C.
Attn: Dr. Leslie G. McCracken, Jr.
Code 3933A

Office of Naval Research
Department of the Navy
Attn: Geophysics Branch, Code 416
Washington 25, D. C.

Office of Chief Signal Officer
Signal Plans and Operations Division
Attn: SIGOL-2, Room 20
Com. Liaison Br., Radio Prop. Sect.
The Pentagon, Washington 25, D. C.

Defence Research Member
Canadian Joint Staff
2001 Connecticut Street
Washington, D. C.

Central Radio Prop. Lab.
National Bureau of Standards
Attn: Technical Reports Library
Boulder, Colorado

U. S. Weather Bureau
U. S. Department of Commerce
Washington 25, D. C.
Attn: Dr. Harry Wexler

Federal Communications Commission
Washington 25, D. C.
Attn: Mrs. Barbara C. Grimes, Librarian

Upper Atmosphere Research Section
Central Radio Propagation Laboratory
National Bureau of Standards
Boulder, Colorado

Argonne National Laboratory
P.O. Box 299
Lemont, Illinois
Attn: Dr. Hoylande D. Young

Bell Telephone Labs.
Murray Hill, New Jersey
Attn: Dr. S. O. Rice, 3B - 203

Carnegie Institute of Washington
Dept. of Terrestrial Magnetism
5241 Broad Branch Road, N. W.
Washington 15, D. C.
Attn: Library

Georgia Tech Research Institute
225 N. Avenue, N. W.
Attn: Dr. James E. Boyd
Atlanta, Georgia

University of Maryland
College Park, Maryland
Attn: Dr. A. Weinstein
Institute of Fluid Dynamics

Massachusetts Institute of Technology
Lincoln Laboratory
P. O. Box 73
Lexington 73, Massachusetts
Attn: Prof. Radford, Division 3 Head

Willow Run Research Center
University of Michigan
Willow Run Airport
Ypsilanti, Michigan
Attn: Dr. C. L. Dolph

School of Engineering
New York University
University Heights
New York, New York

Shell Fellowship Committee of the
Shell Companies Foundation, Inc.
50 West 50th Street
New York 20, N. Y.
Attn: Mr. J. R. Janssen

Esso Research and Engineering Co.
P. O. Box 51
Linden, New Jersey
Attn: Mr. C. L. Brown, Manager

Union Carbide and Carbon Corp.
30 E. 42nd Street
New York 17, New York
Attn: Mr. L. E. Erlandson

Convair
San Diego 12, California
Attn: Mr. Marvin Stern

Bell Telephone Labs., Inc.
463 West Street
New York 13, N. Y.
Attn: Dr. Mervin J. Kelly

Engineering Library
University of California
405 Hilgard Avenue
Los Angeles 24, California

Convair, A Division of General Dynamics Corp.
Daingerfield, Texas
Attn: J. E. Arnold, Division Manager

Convair, A Division of General Dynamics Corp.
San Diego 12, California
Attn: R. L. Bayless, Chief Engineer

Convair, a Division of General Dynamics Corp.
San Diego 12, California
Attn: K. J. Bossart, Chief Engineer-WS107A

Convair, A Div. of General Dynamics Corp.
Fort Worth 1, Texas
Attn: F. W. Davis, Chief Engineer

Convair, A Div. of General Dynamics Corp.
Pomona, California
Attn: C. D. Perrine
Ass't Div. Manager, Engin.

Shell Development Company
Exploration and Production Res. Div.
3737 Bellaire Boulevard
Houston 25, Texas
Attn: Miss Aphrodite Mamoulides

RCA Laboratories
Princeton, New Jersey
Attn: Dr. Charles Polk

Stanford Research Institute
S. Pasadena, California
Attn: Dr. J. Brandstatten

Wayne State University
Kresge-Hooker Science Library
5250 Second Boulevard
Detroit 2, Michigan

ARRA
1 Bond Street
Westbury, L. I., New York
Attn: Dr. Norman Spector

Varian Associates
611 Hansen Way
Palo Alto, California
Attn: Mrs. Perry Conway
Technical Librarian

Case Institute of Technology
Department of Electrical Engin.
University Circle
Cleveland 6, Ohio
Attn: Prof. Robert Plonsey

Dr. Ming S. Wong, CRRKP
Air Force Cambridge Research Center
Laurence G. Hanscom Field
Bedford, Massachusetts

Physics Section
AVCO-RAD Division
20 South Union Street
Lawrence, Mass.
Attn: Dr. Ernest Bauer

Advanced Development Section
Western Development Labs.
Philco Corp.
3875 Fabian Way
Palo Alto, California
Attn: Dr. Albert R. Giddis

Department of Aeronautical Engineering
University of Michigan
Ann Arbor, Michigan
Attn: Prof. Mahinder Uberoi

Gordon McKay Laboratory
Harvard University
Cambridge 39, Mass.
Attn: Dr. S. R. Seshadri

Commander
Air Research and Development Command
Attn: RDTR
Andrews Air Force Base
Washington 25, D. C.

College of Engineering
Dept. of Electrical Engineering
University of Florida
Gainesville, Florida
Attn: Dr. Sullivan

Dr. V. M. Papadopoulos
Dept. of Engineering
Brown University
Providence, R. I.

Major Vernon Lee Dawson
RSDD-OML(MO)
Redstone Arsenal
Huntsville, Alabama

Grumman Aircarft Engineering Corp.
South Oyster Bay Road
Bethpage, Long Island, N. Y.
Attn: Dr. Charles Mack

AF Office of Scientific Research
Washington 25, D. C.
Attn: Dr. Karl Kaplan

University of California
Radiation Laboratory
P. O. Box 808
Livermore, California
Attn: Dr. Bernard A. Lippmann

Department of Electrical Engineering
Case Institute of Technology
University Circle
Cleveland 6, Ohio
Attn: Professor Albert E. Collin

Antenna Laboratory
Air Force Cambridge Research Center
Laurence G. Hanscom Field
Bedford, Massachusetts
Attn: Mr. Philip Blackstone

Lt. Mark J. Beran, CRRD
Air Force Cambridge Research Center
Laurence G. Hanscom Field
Bedford, Massachusetts

Mr. Richard Mack, CRRD
Air Force Cambridge Research Center
Laurence G. Hanscom Field
Bedford, Massachusetts

System Development Corporation
2500 Colorado Avenue
Santa Monica, California
Attn: Library

N. Y. U. Institute of
Mathematical Sciences
25 Waverly Place
New York 3, N. Y.